# A BASIC STRICHARTZ ESTIMATE FOR THE SCHRÖDINGER EQUATION

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## 1. Introduction

We will consider the linear homogeneous Schrödinger equation on $\mathbb{R}^n \times [0, \infty)$ given by $\partial_t u = i\Delta u$, subject to the initial condition $u(x, 0) = u_0(x)$. A solution is usually denoted by $u(x, t) = e^{it\Delta}u_0(x)$. If $2 \leq q, r \leq \infty$, $2/q + n/r = n/2$, and $(q, r, n) \neq (2, \infty, 2)$, then

$$\|e^{it\Delta}u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^n)},$$

where

$$\|u\|_{L_t^q L_x^r} = \left(\int_{\mathbb{R}} \|u(\cdot, t)\|_{L_x^r(\mathbb{R}^n)}^q \, dt\right)^{1/q}, \quad \|u\|_{L_{x,t}^q} = \|u\|_{L_t^q L_x^q}$$

and $f \lesssim g$ denotes the existence of some constant $C$ (here dependent on $q, r, n$) such that $|f| \leq C|g|$. This is the general form of homogeneous Strichartz estimates for the linear Schrödinger equation. Strichartz originally showed such an estimate holds in the case $q = r = 2(n+2)/n$ for the non-homogeneous problem $\partial_t u = i(\Delta u + g)$ (see [4, Corollary 1]):

**Theorem 1.1** (Strichartz, 1977). *If $u = e^{it\Delta}u_0$ solves $\partial_t u = i(\Delta u + g)$ with $u_0 \in L_x^2(\mathbb{R}^n)$ and $g \in L_{x,t}^p(\mathbb{R}^n \times \mathbb{R})$ where $p = 2(n+2)/(n+4) = q'$, then*

$$\|u\|_{L_{x,t}^q} \lesssim (\|u_0\|_{L_x^2(\mathbb{R}^n)} + \|g\|_{L_{x,t}^p}).$$

Here, we will focus our attention on this case with $g = 0$ using a proof presented in lectures by Larry Guth [2, Lectures 29–30]. For simplicity of the argument, we will also suppose $u_0 \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space.

Broadly speaking, these mixed Lebesgue norm estimates are of importance because of how they can provide a quantitative measure of the dispersion of solutions to various PDEs. Moreover, they have applications to problems of interest in the field, such as showing well-posedness for various equations (including nonlinear Schrödinger equations).

### 1.1. Fourier transform.

The convention for the Fourier transform here is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi\cdot x} \, dx.$$

With this choice, Plancherel's theorem reads $\|\hat{f}\|_{L^2_\xi(\mathbb{R})} = \|f\|_{L^2_x(\mathbb{R})}$ and the normalization for Fourier inversion is

$$f(x) = (\hat{f})^\vee(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i \xi \cdot x}\, d\xi.$$

Moreover, the Fourier transform interacts with differentiation as follows. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ denote a multi-index, and define

$$x^\alpha = \prod_{i=1}^n x_i^{\alpha_i} \quad \text{and} \quad \partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}.$$

Then, under suitable conditions,

$$\partial^\alpha \hat{f} = [(-2\pi i x)^\alpha f]\widehat{\ } \quad \text{and} \quad (\partial^\alpha f)\widehat{\ }(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi).$$

## 2. Some preliminaries

We start by performing some formal computations without any conditions on the initial condition $u(x,0) = u_0(x)$. Taking Fourier transforms on $\partial_t u = i\Delta u$, we obtain

$$\partial_t \hat{u}(\xi, t) = i(-4\pi|\xi|^2)\hat{u}(\xi, t).$$

Solving this ODE and using the initial condition $\hat{u}(\xi, 0) = \hat{u}_0(\xi)$, we get

$$\hat{u}(\xi, t) = e^{-4\pi i |\xi|^2 t} \hat{u}_0(\xi).$$

Then, after applying Fourier inversion we recover

$$u(x,t) = [e^{-4\pi i|\xi|^2 t}\hat{u}_0]^\vee(x) = e^{it\Delta} u_0(x).$$

From this process, we see that a desirable condition is $u_0 \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space, as then each step becomes valid, and we note that this implies our computed solution $u \in \mathcal{S}(\mathbb{R}^n)$. From now on, we always assume $u_0 \in \mathcal{S}(\mathbb{R}^n)$, so the derivation of the solution above applies.

### 2.1. Basic estimates.

To obtain our desired Strichartz estimate, we will need some "endpoint estimates" followed by an estimate obtained by interpolation. We begin with the following conservation property.

**Proposition 2.1.** *The $L^2$ norm of a solution is preserved in $t$:*

$$\|e^{it\Delta} u_0\|_{L^2_x(\mathbb{R}^n)} = \|u_0\|_{L^2_x(\mathbb{R}^n)}.$$

*Proof.* This follows from applying Plancherel's theorem twice:

$$\begin{aligned}\|e^{it\Delta} u_0\|_{L^2_x(\mathbb{R}^n)} &= \|[e^{-4\pi i|\xi|^2 t}\hat{u}_0]^\vee\|_{L^2_x(\mathbb{R}^n)} = \|e^{-4\pi i|\xi|^2 t}\hat{u}_0\|_{L^2_\xi(\mathbb{R}^n)} \\ &= \|\hat{u}_0\|_{L^2_\xi(\mathbb{R}^n)} = \|u_0\|_{L^2_x(\mathbb{R}^n)}.\end{aligned}$$

□

**Remark 2.2.** One can show this holds for derivatives as well: $\|\partial_x^\alpha(e^{it\Delta}u_0)\|_{L^2_x(\mathbb{R}^n)}$ is constant in $t$.

**Remark 2.3.** There is no need for the initial data to be Schwartz here, and one can prove this without Fourier analysis. To do this, take real parts after integrating $(\partial_t u - i\Delta u)\overline{u} = 0$.

The following estimate gives decay for solutions to our Schrödinger equation.

**Proposition 2.4.** *We have the dispersion estimate*

$$\|e^{it\Delta}u_0\|_{L_x^\infty(\mathbb{R}^n)} \lesssim t^{-n/2}\|u_0\|_{L_x^1(\mathbb{R}^n)}.$$

*Proof.* By Fourier inversion and dominated convergence, we get

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} \hat{u}(\xi,t)e^{2\pi i\xi\cdot x}\,d\xi = \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2}\hat{u}(\xi,t)e^{2\pi i\xi\cdot x}\,d\xi \\ &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2}e^{-4\pi i|\xi|^2 t}\hat{u}_0(\xi)e^{2\pi i\xi\cdot x}\,d\xi \\ &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^n}\left(e^{-\epsilon|\xi|^2}e^{-4\pi i|\xi|^2 t}e^{2\pi i\xi\cdot x}\int_{\mathbb{R}^n} u_0(y)e^{-2\pi i\xi\cdot y}\,dy\right)d\xi. \end{aligned}$$

Using Fubini,

$$u(x,t) = \lim_{\epsilon\to 0} K_{t,\epsilon} * u_0 = \lim_{\epsilon\to 0}\int_{\mathbb{R}^n} K_{t,\epsilon}(x-y)u_0(y)\,dy$$

where

$$K_{t,\epsilon}(z) = \int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2}e^{-4\pi i|\xi|^2 t}e^{2\pi i\xi\cdot z}\,d\xi,$$

so we have

$$\|e^{it\Delta}u_0(x)\|_{L_x^\infty(\mathbb{R}^n)} \leq \|u_0(x)\|_{L_x^1(\mathbb{R}^n)}\liminf_{\epsilon\to 0}\|K_{t,\epsilon}(z)\|_{L_z^\infty(\mathbb{R}^n)}.$$

To produce estimates of $K_{t,\epsilon}$, notice that

$$\int_{\mathbb{R}^n} e^{-\epsilon|\xi|^2}e^{-4\pi i|\xi|^2 t}e^{2\pi i\xi\cdot z}\,d\xi = \prod_{j=1}^n\int_{\mathbb{R}} e^{-\epsilon\xi_j^2}e^{-4\pi i\xi_j^2 t}e^{2\pi i\xi_j z_j}\,d\xi_j.$$

One checks (either by using the Gaussian formula for $e^{-a\xi^2}$ using $a$ having positive real part, or splitting the integral appropriately to estimate) that

$$\left|\int_{\mathbb{R}} e^{-\epsilon\xi_j^2}e^{-4\pi i\xi_j^2 t}e^{2\pi i\xi_j z_j}\,d\xi_j\right| \lesssim t^{-1/2}$$

uniformly as $\epsilon\to 0$, so we are done. □

**Remark 2.5.** One can explicitly compute the Schrödinger kernel

$$K_t(z) = \frac{1}{(4\pi it)^{n/2}}e^{i\frac{|z|^2}{4t}}$$

which appeared in this process, and the solution to our PDE would then be given by $u = K_t * u_0$. This way, the dispersion estimate is easily obtained.

**Remark 2.6.** This estimate is sharp. If $u_0$ is a smooth bump supported on the unit ball $B_1$ in $\mathbb{R}^n$, then the above computation can be used to show that

$$\|e^{it\Delta}u_0\|_{L^\infty_x(\mathbb{R}^n)} \gtrsim t^{-n/2}\|u_0\|_{L^1_x(\mathbb{R}^n)}.$$

Using the preservation of the $L^2$ norm and this dispersion estimate, we can interpolate to obtain $L^p_x(\mathbb{R}^n) \to L^{p'}_x(\mathbb{R}^n)$ estimates in a wider range. First, we recall Riesz–Thorin interpolation theorem:

**Theorem 2.7** (Riesz–Thorin interpolation)**.** *Let $X$ and $Y$ be $\sigma$-finite measure spaces (under some implicit measure) and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Suppose $T : L^{p_0}(x) + L^{p_1}(x) \to L^{q_0}(Y) + L^{q_1}(Y)$ is a linear operator that is bounded as an operator $L^{p_0}(X) \to L^{q_0}(Y)$ and $L^{p_1}(X) \to L^{q_1}(X)$. Then, for all $\theta \in (0,1)$, we have that $T$ is bounded as a map $L^{p_\theta}(X) \to L^{q_\theta}(Y)$ where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

*Moreover,*

$$\|T\|_{L^{p_\theta}(X)\to L^{q_\theta}(Y)} \leq \|T\|^{1-\theta}_{L^{p_0}(X)\to L^{q_0}(Y)}\|T\|^{\theta}_{L^{p_1}(X)\to L^{q_1}(X)}.$$

Applying Riesz–Thorin gives the following estimate.

**Corollary 2.7.1.** *Let $1/p = (1-\theta)/2$. Then,*

$$\|e^{it\Delta}u_0\|_{L^p_x(\mathbb{R}^n)} \lesssim t^{-n\theta/2}\|u_0\|_{L^{p'}_x(\mathbb{R}^n)}.$$

*Proof.* We have the endpoint estimates $\|e^{it\Delta}u_0\|_{L^2_x(\mathbb{R}^n)} \leq \|u_0\|_{L^2_x(\mathbb{R}^n)}$ for $p_0 = q_0 = 2$ and $\|e^{it\Delta}u_0\|_{L^\infty_x(\mathbb{R}^n)} \lesssim t^{-n/2}\|u_0\|_{L^1_x(\mathbb{R}^n)}$ for $p_1 = 1, q_1 = \infty$. For $\theta \in (0,1)$, the Riesz–Thorin exponent $p$ between $p_1$ and $q_1$ satisfies

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{\infty} = \frac{1-\theta}{2},$$

and the $L^2_x(\mathbb{R}^n) \to L^2_x(\mathbb{R}^n)$ operator norm of $e^{it\Delta}$ is 1, so

$$\|e^{it\Delta}\|_{L^{p'}_x(\mathbb{R}^n)\to L^p_x(\mathbb{R}^n)} \leq \|e^{it\Delta}\|^{\theta}_{L^1_x(\mathbb{R}^n)\to L^\infty_x(\mathbb{R}^n)} \lesssim (t^{-n/2})^\theta = t^{-n\theta/2}.$$

This is precisely the operator bound we sought.

As a note of caution, we remark that the implicit constant here has dependence on $\theta$. ∎

2.2. **Properties of $e^{it\Delta}$.** We note two facts we will later use about the operator $e^{it\Delta}$ below.

**Lemma 2.7.2.** *$e^{it\Delta}$ defines a unitary operator on $L^2_x(\mathbb{R}^n)$.*

*Proof.* Compute

$$\begin{aligned}\langle e^{it\Delta} f, g\rangle &= \int_{\mathbb{R}^n} [e^{it\Delta} f](x)\overline{g(x)}\,dx = \int_{\mathbb{R}^n} [e^{-4\pi i|\xi|^2 t}\hat{f}(\xi)]^\vee(x)\overline{g(x)}\,dx \\ &= \int_{\mathbb{R}^n} e^{-4\pi i|\xi|^2 t}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi = \int_{\mathbb{R}^n} \hat{f}(\xi)(\overline{e^{4\pi i|\xi|^2 t}\hat{g}(\xi)})\,d\xi \\ &= \int_{\mathbb{R}^n} f(x)\overline{[e^{4\pi i|\xi|^2 t}\hat{g}(\xi)]^\vee(x)}\,dx = \int_{\mathbb{R}^n} f(x)\overline{[e^{-it\Delta} g](x)}\,dx \\ &= \langle f, e^{-it\Delta} g\rangle.\end{aligned}$$

using the unitarity of the Fourier transform on $L^2(\mathbb{R}^n)$ $\square$

**Lemma 2.7.3.** $e^{it\Delta}$ *has the semigroup property*

$$e^{is\Delta}(e^{it\Delta}u_0) = e^{i(s+t)\Delta}u_0.$$

*Proof.* We have

$$\begin{aligned}e^{is\Delta}(e^{it\Delta}u_0) &= [e^{-4\pi i|\xi|^2 s}(e^{it\Delta}u_0)^\wedge]^\vee = [e^{-4\pi i|\xi|^2 s}(e^{-4\pi i|\xi|^2 t}u_0)]^\vee \\ &= [e^{-4\pi i|\xi|^2 (s+t)}u_0]^\vee = e^{i(s+t)\Delta}u_0\end{aligned}$$

by Fourier inversion. $\square$

## 3. Proof of Strichartz estimates

We prove the Strichartz estimate

$$\|e^{it\Delta}u_0\|_{L^q_{x,t}} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^n)} \tag{1}$$

with $q = 2(n+2)/n$ in this section. First, we start with a Duhamel principle formula for the inhomogeneous Schrödinger equation $\partial_t u = i\Delta u + f$. Then, we obtain a precursory Strichartz estimate using Hardy–Littlewood–Sobolev fractional integral theory, and afterwards are able to conclude the estimate (1) with the help of a duality trick.

**Proposition 3.1** (Duhamel principle)**.** *Suppose* $f \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$. *Then,*

$$u(x,t) = \int_{-\infty}^t e^{i(t-s)\Delta} f(x,s)\,ds$$

*solves* $\partial_t u = i\Delta u + f$. *Moreover, let* $T$ *be such that* $(x,t) \notin \operatorname{supp} f$ *for all* $t < T$. *Then, for all* $t < T$, $u(x,t) = 0$ *(we then say that* $u$ *vanishes before the support of* $f$*).*

*Proof.* Let $u$ be as defined above. Applying the Leibniz integral rule gives

$$\begin{aligned}\partial_t u(x,t) &= e^{i(t-s)\Delta} f(x,s)|_{s=t} + \int_{-\infty}^{x} \partial_t (e^{i(t-s)\Delta} f(x,s))\, ds \\ &= f(x,t) + \int_{-\infty}^{x} i\Delta (e^{i(t-s)\Delta} f(x,s))\, ds \\ &= f(x,t) + i\Delta \left( \int_{-\infty}^{x} e^{i(t-s)\Delta} f(x,s)\, ds \right) \\ &= i\Delta u(x,t) + f(x,t).\end{aligned}$$

Then we see that for $t < T$, $\partial_t u = 0$, and $u$ is Schwartz so we conclude that $u(x,t) = 0$ for all $(x,t) \in \mathbb{R}^n \times (-\infty, T)$. □

Using the Hardy–Littlewood–Sobolev fractional integral theorem stated below, we will obtain a precursory Strichartz estimate.

**Theorem 3.2** (Hardy–Littlewood–Sobolev fractional integration)**.** *Let $\alpha$ be such that $0 < \alpha < n$. If $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then the operator $f \mapsto K_\alpha * f$ where $K_\alpha(x) = |x|^{\alpha - n}$ is $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ bounded.*

**Lemma 3.2.1** (Strichartz estimate, Duhamel solution)**.** *Let $f \in C_c^\infty(\mathbb{R})$ and let $u$ satisfy $\partial_t u = i\Delta u + f$. Also suppose that $u$ vanishes before the support of $f$. Then,*

$$\|u\|_{L^q_{x,t}} \lesssim \|f\|_{L^{q'}_{x,t}}.$$

*Proof.* Minkowski gives

$$\|u(\cdot,t)\|_{L^q_x(R^n)} \leq \int_{-\infty}^{\infty} \|e^{i(t-s)\Delta} f(x,s)\|_{L^q_x(\mathbb{R}^n)}\, ds.$$

Taking $\theta = 1 - \frac{2}{q}$, we can apply Corollary 2.7.1 and obtain

$$\|u(\cdot,t)\|_{L^q_x(\mathbb{R}^n)} \lesssim \int_{-\infty}^{\infty} (t-s)^{-n\theta/2} \|f(\cdot,s)\|_{L^{q'}_x(\mathbb{R}^n)}\, ds$$

which is a fractional Hardy–Littlewood–Sobolev fractional integral. One checks with our choice of $\theta$, recalling $q = 2(n+2)/n$, that the conditions for the Hardy–Littlewood–Sobolev fractional integration theorem are met, and applying it gives the desired estimate. □

With this, we now prove the Strichartz estimate (1):

**Theorem 3.3.** *If $u_0 \in \mathcal{S}(\mathbb{R}^n)$ and $u$ solves the homogeneous Schrödinger problem*

$$\begin{cases} \partial_t u(x,t) = i\Delta u(x,t) & \text{for } t > 0, \\ u(x,0) = u_0(x), \end{cases}$$

*then*

$$\|e^{it\Delta} u_0\|_{L^q_{x,t}} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^n)} \tag{1}$$

*where $q = 2(n+2)/n$.*

*Proof.* Let $q' = 2(n+2)/(n+4)$ be the $L^q$ dual exponent $(1/q + 1/q' = 1)$, so duality gives

$$\|e^{it\Delta}u_0\|_{L^q_{x,t}} = \sup_{\|f\|_{L^{q'}_{x,t}}=1} \int_{\mathbb{R}} \int_{\mathbb{R}^n} [e^{it\Delta}u_0](x)\overline{f(x,t)}\,dx\,dt,$$

and since $e^{it\Delta}$ is unitary this is equal to

$$\sup_{\|f\|_{L^{q'}_{x,t}=1}} \int_{\mathbb{R}} \langle u_0, e^{-it\Delta}f(\cdot,t)\rangle\,dt = \sup_{\|f\|_{L^{q'}_{x,t}}=1} \langle u_0, \int_{\mathbb{R}} e^{-it\Delta}f(\cdot,t)\,dt\rangle.$$

Here, we may apply Cauchy–Schwarz to obtain

$$\|e^{it\Delta}u_0\|_{L^q_{x,t}} \le \|u_0\|_{L^2_x(\mathbb{R}^n)} \sup_{\|f\|_{L^{q'}_{x,t}}=1} \|\int_{\mathbb{R}} e^{-it\Delta}f(\cdot,t)\,dt\|_{L^2_x(\mathbb{R}^n)}. \tag{2}$$

Now, observe that

$$\begin{aligned}
\|\int_{\mathbb{R}} e^{-it\Delta}f(\cdot,t)\,dt\|^2_{L^2_x(\mathbb{R}^n)} &= \langle \int_{\mathbb{R}} e^{-it\Delta}f(\cdot,t)\,dt, \int_{\mathbb{R}} e^{-is\Delta}f(\cdot,s)\,ds\rangle \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \langle e^{-it\Delta}f(\cdot,t), e^{-is\Delta}f(\cdot,s)\,\rangle\,dt\,ds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \langle f(\cdot,t), e^{i(t-s)\Delta}f(\cdot,s)\rangle\,dt\,ds \\
&= \int_{\mathbb{R}} \langle f(\cdot,t), \int_{\mathbb{R}} e^{i(t-s)\Delta}f(\cdot,s)\,ds\rangle\,dt
\end{aligned}$$

using the semigroup property. Next, writing

$$F(x,t) = \int_{\mathbb{R}} e^{i(t-s)\Delta}f(x,s)\,ds,$$

we have

$$\begin{aligned}
\|\int_{\mathbb{R}} e^{-it\Delta}f(\cdot,t)\,dt\|^2_{L^2_x(\mathbb{R}^n)} &= \int_{\mathbb{R}} \langle f(\cdot,t), F(\cdot,t)\rangle\,dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^n} f(x,t)\overline{F(x,t)}\,dx\,dt.
\end{aligned}$$

Using Hölder's inequality for $L_{x,t}$ norms, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} f(x,t)\overline{F(x,t)}\,dx\,dt \le \|f\|_{L^{q'}_{x,t}}\|F\|_{L^q_{x,t}}.$$

But $F$ is a solution for which the Duhamel solution estimate Lemma 3.2.1 applies to get $\|F\|_{L^q_{x,t}} \lesssim \|f\|_{L^{q'}_{x,t}}$, so by (2) we get

$$\|e^{it\Delta}u_0\|_{L^q_{x,t}} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^n)} \sup_{\|f\|_{L^{q'}_{x,t}}=1} \left(\|f\|^2_{L^{q'}_{x,t}}\right)^{1/2} = \|u_0\|_{L^2_x(\mathbb{R}^n)}$$

which is our Strichartz estimate. $\square$

3.1. **Comments on $TT^*$ methods.** The proof of (1) can be seen as an application of what is known as the $TT^*$ method (or $T^*T$ method). This method, used in the study of dispersive PDE, revolves around the following result (see [1, Lemma 2.2]).

**Lemma 3.3.1.** *Let $X$ be a Hilbert space and $Y$ a Banach space. Then, if one of $T : X \to Y$, $T^* : Y^* \to X$, or $TT^* : Y^* \to Y$ is a bounded operator, then all three are.*

Put $T(u) = e^{it\Delta}u$ where $T : L^2_x(\mathbb{R}^n) \to L^q_{x,t}(\mathbb{R}^n \times \mathbb{R})$. Then, we see that $T^*(f) : L^{q'}_{x,t}(\mathbb{R}^n \times \mathbb{R}) \to L^2_x(\mathbb{R}^n)$ is given by

$$T^*(f) = \int_{\mathbb{R}} e^{-it\Delta} f(x,t)\, dt$$

and thus $TT^* : L^{q'}_{x,t}(\mathbb{R}^n \times \mathbb{R}) \to L^q_{x,t}(\mathbb{R}^n \times \mathbb{R})$ is given by

$$TT^*(g) = \int_{\mathbb{R}} e^{i(t-s)\Delta} f(x,s)\, ds.$$

In the preceding proof, we obtained the bound

$$\| \int_{\mathbb{R}} e^{-it\Delta} f(\cdot, t) \|_{L^2_x(\mathbb{R}^n)} \lesssim \|f\|_{L^{q'}_{x,t}},$$

showing that $T^*$ is a bounded operator; by the $TT^*$ method this implies bounds on the other two operators $T$ and $TT^*$, with the boundedness of $T$ being our Strichartz estimate (1).

More generally, $TT^*$ methods come in various forms and are ubiquitous throughout harmonic analysis. For some history on their usage in different contexts of harmonic analysis, the reader is referred to [3, Section 1.1].

## References

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