SHIMURA LIFTS OF NEARLY HOLOMORPHIC MODULAR FORMS

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Abstract. We extend the Shimura lifts to nearly holomorphic modular forms of half-integral weight. Then, we show Hecke equivariance for this extension of the Shimura lifts, as well as new Selberg identities in this context.

1. Introduction

Let $f(z)$ be a function on the upper half plane $\mathbb{H}$ with $z = x + iy$. For an integer $k \geq 0$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(4)$ we denote the weight $k + 1/2$ double slash operator [12, p. 447]

$$\left(f||_{k+1/2}\gamma\right)(z) := \left(\frac{c}{d}\right)\varepsilon_d^{2k+1}(cz+d)^{-k-1/2}f(\gamma z). \tag{1.1}$$

Here $\left(\frac{c}{d}\right)$ denotes the Kronecker symbol and $\varepsilon_d = \left(\frac{-4}{d}\right)^{1/2}$ for an odd integer d is given by

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

If l is a real number and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ we also define the weight l single slash operator as

$$\left(f|_l\gamma\right)(z) := \det(\gamma)^{l/2}(cz+d)^{-l}f(\gamma z), \tag{1.2}$$

see Cohen [4, p. 280]. Here and above, $(cz+d)^l$ is taken to be the principal branch when $l \notin \mathbb{Z}$.

A holomorphic function $f(z)$ on the upper plane $\mathbb{H}$ is called a modular form of weight $k+1/2$, denoted $f \in M_{k+1/2}(4)$, if for every $\gamma \in \Gamma_0(4)$

$$f||_{k+1/2}\gamma = f,$$

and f is holomorphic at the cusps of $\Gamma_0(4)$; see Shimura [12]. We say $f \in M_{k+1/2}(4)$ is a cusp form, denoted $f \in S_{k+1/2}(4)$, if f vanishes at the cusps of $\Gamma_0(4)$. Analogously, for an even integer $k \geq 4$, we write $M_k(1)$ (resp. $S_k(1)$) for the space of modular forms (resp. cusp forms) of weight k and level one.

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In order to state and prove cleaner results we shall confine ourselves to the Kohnen's plus subspace of $M_{k+1/2}(4)$, defined by

$$M^+_{k+1/2}(4) := \{f(z) = \sum_{n\geq 0} a_f(n)q^n \in M_{k+1/2}(4) \mid a_f(n) = 0 \text{ if } (-1)^k n \equiv 2,3 \pmod 4\},$$

where $q = e^{2\pi i z}$, and we denote $S^+_{k+1/2}(4) = M^+_{k+1/2} \cap S_{k+1/2}(4)$.

Let D be a fundamental discriminant (i.e. $D = 1$ or is the discriminant of a quadratic field) such that $(-1)^k D > 0$. Following Kohnen [7, p. 251], for $f(z) = \sum_{n\geq 0} a_f(n)q^n \in M^+_{k+1/2}(4)$, we define its D-th Shimura lift as

$$\mathcal{S}_D\left(\sum_{n\geq 0} a_f(n)q^n\right) := \frac{a_f(0)}{2}L_D(1-k) + \sum_{n\geq 1}\left(\sum_{d\mid n}\left(\frac{D}{d}\right)d^{k-1}a_f\left(|D|\frac{n^2}{d^2}\right)\right)q^n, \tag{1.3}$$

where $\left(\frac{D}{\cdot}\right)$ as before is the Kronecker symbol and $L_D(s)$ is its associated L-function. It is known that $\mathcal{S}_D$ maps $M^+_{k+1/2}(4)$ to $M_{2k}(1)$ and $S^+_{k+1/2}(4)$ to $S_{2k}(1)$, and commutes with the action of Hecke operators; see Kohnen [7, Theorem 1] and Shimura [12].

In this context, we recall a beautiful identity for the Shimura lift $\mathcal{S}_1$, first observed by Selberg. Suppose $f(z) = \sum_{n\geq 0} a_f(n)q^n \in M_k(1)$ is a normalized Hecke eigenform. Here, the normalization on f is made such that $a_f(1) = 1$. In particular, if f is noncuspidal, then $f(z) = G_k(z) = \zeta(1-k)/2 + \sum_{n\geq 1}\sigma_{k-1}(n)q^n$. Let $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2} \in M^+_{1/2}$ be the classic theta function. Then

$$\mathcal{S}_1(f(4z)\theta(z)) = f(z)^2 \in M_{2k}. \tag{1.4}$$

This identity has been recently generalized to Rankin-Cohen brackets of modular forms in [3] and [16], independently. Let $f(z)$ and $g(z)$ be two smooth functions on $\mathbb{C}$. For any given real numbers a, b and an integer $e \geq 0$, the e-th Rankin-Cohen bracket of $f(z)$ and $g(z)$ is defined as [18, (1)]:

$$[f(z), g(z)]^{(a,b)}_e = \sum_{r=0}^{e}(-1)^r\binom{e+a-1}{e-r}\binom{e+b-1}{r}f(z)^{(r)}g(z)^{(e-r)},$$

where $f(z)^{(r)} := \frac{1}{(2\pi i)^r}\frac{d^r f(z)}{dz^r}$ is the normalized r-th derivative of f with respect to z. When a and b are clear from context, such as by being weights of (nearly holomorphic) modular forms, we may choose to omit (a, b) from the notation. Here, the binomial coefficient $\binom{a}{b}$ for complex a, b is defined through gamma functions as $\frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$.

Choie-Kohnen-Zhang [3, Proposition 2.1] and Xue [16, Theorem 1.1] independently showed that if $k \geq 4$ is an even integer, $f(z) \in M_k(1)$ is a normalized Hecke eigenform, and e is a nonnegative integer, then

$$\mathcal{S}_1\left([f(4z), \theta(z)]_e\right) = \frac{\binom{k+e-1}{e}}{\binom{k+2e-1}{2e}}[f(z), f(z)]_{2e}. \tag{1.5}$$

Later, a similar Selberg identity has been obtained in [5] for the Shimura lift $\mathcal{S}_D$ with D odd; see Section 5 for the detail.

For a real number l and integer $n \geq 0$, the Maass-Shimura operator of weight l is given by

$$\delta_l^{(n)} = \left(\frac{1}{2\pi i}\right)^n \left(\frac{l+2n-2}{2iy} + \frac{d}{dz}\right)\left(\frac{l+2n-4}{2iy} + \frac{d}{dz}\right)\cdots\left(\frac{l}{2iy} + \frac{d}{dz}\right),$$

where $\delta_l^{(0)}$ is the identity operator and we write $\delta_l^{(1)} = \delta_l$. Recall that for k an integer, if a function $f(z) = \sum_{j=0}^n c_j y^{-j} f_j(z)$ on $\mathbb{H}$ satisfies for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$(f|_k\gamma)(z) = f(z),$$

and such that each f_j is holomorphic with a holomorphic Fourier expansion at ∞, then $f(z)$ is called a nearly holomorphic modular form of weight k in the sense of Shimura [13], denoted $f(z) \in \widehat{M}_k(1)$. It is well known that $\widehat{M}_k(1)$ is spanned by nearly holomorphic forms like $\delta_{k-2n}^{(n)}(g)$ for $g \in M_{k-2n}(1)$; see Shimura [13, Lemma 7] or Lanphier [8, Lemma 1].

We now define the notion of nearly holomorphic modular forms of half-integral weight.

Definition 1.1. Let $k \geq 0, n \geq 0$ be integers. A smooth function $f(z)$ on the upper half plane is called a nearly holomorphic modular form of weight $k+1/2$ and depth n provided that

(1) $f||_{k+1/2}\gamma = f$ for all $\gamma \in \Gamma_0(4)$, and
(2) for each $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ we have $(f|_{k+1/2}\alpha)(z) = \sum_{j=0}^n y^{-j} f_{\alpha,j}(z)$ with $f_{\alpha,j}(z)$ holomorphic, and $f_{\alpha,j}(z) = \sum_{m=0}^\infty a_{\alpha,j}(n)q^n$, that is $f_{\alpha,j}$ is holomorphic at infinity.

We denote the space generated by nearly holomorphic modular forms of weight $k+1/2$ (and any depth) as $\widehat{M}_{k+1/2}(4)$. If $f(z) \in \widehat{M}_{k+1/2}(4)$ furthermore satisfies for $\alpha = I \in \mathrm{SL}_2(\mathbb{Z})$ and all j

(3) $a_{I,j}(m) = 0$ if $(-1)^k m \equiv 2, 3 \pmod 4$,

then we denote $f(z) \in \widehat{M}^+_{k+1/2}(4)$.

Similar to the integral weight case, we will show that the space $\widehat{M}^+_{k+1/2}(4)$ is actually generated by $\delta_{k-2n+1/2}^{(n)}(f)$ for $f \in M^+_{k-2n+1/2}(4)$ and $n \geq 0$; see Lemma 2.1. For an integer $k \geq 0$, it is not hard to verify that $\delta_k^{(n)} : \widehat{M}_k(1) \to \widehat{M}_{k+2n}(1)$ and $\delta_{k+1/2}^{(n)} : \widehat{M}^+_{k+1/2}(4) \to \widehat{M}^+_{k+2n+1/2}(4)$.

Let D be a fundamental discriminant such that $(-1)^k D > 0$. We extend the D-th Shimura lift (1.3) on $M^+_{k+1/2}(4)$ to $\widehat{M}^+_{k+1/2}(4)$ by letting

$$\mathcal{S}_D\left(\delta_{k-2n+1/2}^{(n)}(f)\right) := |D|^n \delta_{2k-4n}^{(2n)}(\mathcal{S}_D(f)), \tag{1.6}$$

where $f \in M^+_{k-2n+1/2}(4)$. One of our main results is the following extended Selberg identity for the first Shimura lift $\mathcal{S}_1$.

Theorem 1.2. Let $f \in M_k(1)$ be a normalized Hecke eigenform of weight $k \geq 4$. Furthermore, let $n, e \geq 0$ be integers. Then,

$$\mathcal{S}_1\left([\delta_k^{(n)}(f(4z)), \theta(z)]_e\right) = 4^n \frac{\binom{k+e+2n-1}{e}}{\binom{k+2e+2n-1}{2e}} [\delta_k^{(n)} f(z), \delta_k^{(n)} f(z)]_{2e}.$$

More generally, if D is an odd fundamental discriminant such that $(-1)^k D > 0$, then we will show in Theorem 5.2 that a similar identity for $\mathcal{S}_D$ also holds true.

The paper is organized as follows. In Section 2 we present several combinatorial identities on Maass-Shimura operators and Rankin-Cohen brackets. Many of these identities have been proven in Lanphier [8] for modular forms of integral weight. However, we need to make sure that they continue to hold for modular forms of half-integral weight. We show that the extended Shimura lifts $\mathcal{S}_D$ (1.6) are Hecke equivariant in Section 3. This property is of critical importance and justifies the validity of our definition of these extended Shimura lifts. Section 4 is devoted to the proof of Theorem 1.2, which will be extended to $\mathcal{S}_D$ for odd D in Section 5. We will discuss in Section 6 the relationship between our results and previous work on the Shimura correspondence.

2. Combinatorial identities on Maass-Shimura operators

In this section we state and prove some necessary combinatorial identities involving Maass-Shimura operators and Rankin-Cohen brackets.

We first present the following result asserting that $\widehat{M}^+_{k+1/2}(4)$ is generated by $\delta^{(n)}_{k-2n+1/2}(f)$ with $f \in M^+_{k-2n+1/2}(4)$ and $n \geq 0$. Our proof closely follows that of Shimura [13, Lemma 7].

Lemma 2.1. Suppose $f \in \widehat{M}^+_{k+1/2}(4)$ is of depth n, that is for $\gamma \in \Gamma_0(4)$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$

$$\left(f||_{k+1/2}\gamma\right)(z) = f(z) \quad \text{and} \quad (f|_{k+1/2}\alpha)(z) = \sum_{j=0}^{n} y^{-j} f_{\alpha,j}(z),$$

where for each j, $f_{\alpha,j}(z)$ is holomorphic on $\mathbb{H}$ and ∞, and $f_{I,j}(z) = \sum_{m\geq 0} a_{I,j}(m)q^m$ such that $a_{I,j}(m) = 0$ if $(-1)^k m \equiv 2, 3 \pmod 4$. Then

$$f(z) = (-4\pi)^n \frac{\Gamma(k-2n+1/2)}{\Gamma(k-n+1/2)} \delta^{(n)}_{k-2n+1/2} f_{I,n}(z) + \sum_{j=0}^{n-1} \delta^{(j)}_{k-2j+1/2} h_j(z)$$

where $h_j(z) \in M^+_{k-2j+1/2}(4)$ and $f_{I,n}(z) \in M^+_{k-2n+1/2}(4)$.

Proof. The $n = 0$ case is immediate, so we assume the claim holds for $n-1$ for some $n \geq 1$. For $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we denote $\mathrm{Im}(\alpha z)$ as $\alpha(y)$, and have $\alpha(y)^{-1} = y^{-1}(cz+d)^2 - 2ci(cz+d)$.

First, we verify that $f_{I,n} \in M^+_{k-2n+1/2}(4)$. Let $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ be as above and notice that

$$\begin{aligned} f(\alpha z) =& \sum_{j=0}^{n} \alpha(y)^{-j} f_{I,j}(\alpha z) = \left(y^{-1}(cz+d)^2 - 2ci(cz+d)\right)^n f_{I,n}(\alpha z) + \sum_{j=0}^{n-1} \alpha(y)^{-j} f_{I,j}(\alpha z) \\ =& f_{I,n}(\alpha z) \sum_{r=0}^{n} \binom{n}{r} y^{-r}(cz+d)^{2r}(-2ci)^{n-r}(cz+d)^{n-r} + \sum_{j=0}^{n-1} \alpha(y)^{-j} f_{I,j}(\alpha z). \end{aligned} \tag{2.1}$$

On the other hand, by our assumption

$$f(\alpha z) = (c+dz)^{k+1/2}(f|_{k+1/2}\alpha)(z) = (c+dz)^{k+1/2}\sum_{j=0}^{n} y^{-j} f_{\alpha,j}(z). \tag{2.2}$$

Comparing the coefficients of y^{-n} in (2.1) and (2.2) we conclude that

$$f_{I,n}(\alpha z)(cz+d)^{2n} = (c+dz)^{k+1/2} f_{\alpha,n}(z), \quad \text{or} \quad (f_{I,n}|_{k-2n+1/2}\alpha)(z) = f_{\alpha,n}(z). \tag{2.3}$$

This means that $f_{I,n}$ is holomorphic at the cusps of $\Gamma_0(4)$. Now, assume $\alpha \in \Gamma_0(4)$. Then $(f||_{k+1/2}\alpha)(z) = f(z)$, and by (1.1) and our assumption

$$f(\alpha z) = \left(\left(\frac{c}{d}\right)\varepsilon_d^{2k+1}(cz+d)^{-k-1/2}\right)^{-1} f(z) = \left(\left(\frac{c}{d}\right)\varepsilon_d^{2k+1}(cz+d)^{-k-1/2}\right)^{-1} \sum_{j=0}^{n} y^{-j} f_{I,j}(z).$$

Comparing the coefficients of y^{-n} in (2.1) and this equation, we get

$$\left(\left(\frac{c}{d}\right)\varepsilon_d^{2k+1}(cz+d)^{-k-1/2}\right)^{-1} f_{I,n}(z) = (cz+d)^{2n} f_{I,n}(\alpha z).$$

Since $\varepsilon_d^{2k+1} = \varepsilon_d^{2k-4n+1}$, we conclude that $f_{I,n}||_{k-2n+1/2}\alpha(z) = f_{I,n}(z)$ for $\alpha \in \Gamma_0(4)$. At last, by our assumption on the Fourier coefficients of $f_{I,n}$ we get $f_{I,n} \in M^+_{k-2n+1/2}(4)$.

Recall that for any smooth function g on $\mathbb{H}$ and real number $l > 0$ [19, (56)]:

$$\delta_l^{(n)} g(z) = (-4\pi)^{-n}\,(l)_n\, y^{-n} g(z) + \sum_{j=0}^{n-1} P_{j,l}^{(n)} (-4\pi)^{-j} y^{-j} g^{(n-j)}(z) \tag{2.4}$$

where $P_{j,l}^{(n)} = \binom{n}{j}(l+n-j)_j$, $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol, and as before $g^{(j)}(z) = \frac{1}{(2\pi i)^j}\frac{d^j g}{dz^j}(z)$ is the j-th normalized derivative.

To complete the induction process define the function $h(z)$ by

$$h(z) = f(z) - (-4\pi)^n \frac{\Gamma(k-2n+1/2)}{\Gamma(k-n+1/2)} \delta^{(n)}_{k-2n+1/2} f_{I,n}(z).$$

It remains to show that $h(z) \in \widehat{M}^+_{k+1/2}(4)$ and is of depth $n-1$. It is clear that $h(z)$ satisfies conditions (1) and (3) of Definition 1.1 because $f \in \widehat{M}^+_{k+1/2}(4)$ and $f_{I,n} \in M^+_{k-2n+1/2}(4)$. By (2.3) and commutativity between Maass-Shimura and slash operators (Cohen [4, Lemma 7.3])

$$\begin{aligned}(h|_{k+1/2}\alpha)(z) &= (f|_{k+1/2}\alpha)(z) - (-4\pi)^n \frac{\Gamma(k-2n+1/2)}{\Gamma(k-n+1/2)} \left(\left(\delta^{(n)}_{k-2n+1/2} f_{I,n}\right)|_{k+1/2}\alpha\right)(z) \\ &= (f|_{k+1/2}\alpha)(z) - (-4\pi)^n \frac{\Gamma(k-2n+1/2)}{\Gamma(k-n+1/2)} \delta^{(n)}_{k-2n+1/2}(f_{\alpha,n}(z)).\end{aligned}$$

Applying (2.4) to $f_{\alpha,n}(z)$ and using the assumption that $(f|_{k+1/2}\alpha)(z) = \sum_{j=0}^{n} y^{-j} f_{\alpha,j}(z)$, we get

$$(h|_{k+1/2}\alpha)(z) = \sum_{j=0}^{n-1} y^{-j} f_{\alpha,j}(z) - (-4\pi)^n \frac{\Gamma(k-2n+1/2)}{\Gamma(k-n+1/2)} \sum_{j=0}^{n-1} P^{(n)}_{j,k-2n+1/2} (-4\pi)^{-j} y^{-j} f_{\alpha,n}^{(n-j)}(z)$$

$$=\sum_{j=0}^{n-1} y^{-j}\left(f_{\alpha,j}(z)-\frac{\Gamma(k-2n+1/2)}{\Gamma(k-n+1/2)}P^{(n)}_{j,k-2n+1/2}(-4\pi)^{n-j}f^{(n-j)}_{\alpha,n}(z)\right).$$

Since $f_{\alpha,j}(z)$ and their derivatives are holomorphic on $\mathbb{H}$ and ∞, $h(z)$ also satisfies condition (2) of Definition 1.1 for $n-1$. This completes the induction argument. □

The following results were shown in Lanphier [8] for modular forms of integral weights k and l. In [17], they were extended to smooth functions (still with integer parameters). We can further extend these to when k, l are positive real numbers using proofs similar to the ones in [17].

Proposition 2.2. Let f and g be smooth functions on the upper half plane. Then for real $k, l > 0$

$$[f,g]_n^{(k,l)}=\sum_{r=0}^{n}\binom{k+n-1}{n-r}\binom{l+n-1}{r}\delta_k^{(r)}(f)\times\delta_l^{(n-r)}(g).$$

Proof sketch. Refer to Xue and Prabath [17, Proposition 2.3] and replace the factorials with gamma functions to show that this holds for real $k, l > 0$. □

Proposition 2.3. Let f and g be smooth functions on the upper half plane. Then for real $k, l > 0$

$$[f,g]_n^{(k,l)}=\sum_{j=0}^{n}a_j(n)\delta^{(n-j)}_{k+l+2j}\left(\delta_k^{(j)}(f)\times g\right), \tag{2.5}$$

where

$$a_j(n)=(-1)^j\binom{k+n-1}{n-j}\binom{k+l+n+j-2}{j}.$$

Proof sketch. Refer to Xue and Prabath [17, Proposition 2.4] and replace the factorials with gamma functions to show that this holds for real $k, l > 0$. Note that the alternating Vandermonde convolution identity [11, p. 11] used in the proof,

$$(-1)^M\binom{N+p}{M+p}=\sum_{j=0}^{M}(-1)^j\binom{M}{j}\binom{N+p+j}{p+j},$$

can be verified when N or p are real numbers by induction on M and Pascal's identity. □

Now, we are able to show that $[\delta_k^{(n)}(f(4z)),\theta(z)]_e$ belongs to $\widehat{M}^+_{k+2n+2e+1/2}(4)$.

Proposition 2.4. Let $f\in M_k(1)$ where $k\geq 4$. Then for $n, e\geq 0$ we have

$$[\delta_k^{(n)}(f(4z)),\theta(z)]_e\in\widehat{M}^+_{k+2n+2e+1/2}(4).$$

Proof. Write $g(z)=f(4z)$. For any integer $m\geq 0$, it is immediate that for $\gamma=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\Gamma_0(4)$

$$\left(\delta_k^{(m)}g\times\theta\right)||_{k+2m+1/2}\gamma(z)=\delta_k^{(m)}(g(z))\times\theta(z).$$

That is $\delta_k^{(m)} g(z) \times \theta(z)$ satisfies condition (1) of Definition 1.1. Now, letting $\alpha \in \mathrm{SL}_2(\mathbb{Z})$ and applying (2.4) to $(g|_k\alpha)(z)$ we see

$$\begin{aligned}\left(\delta_k^{(m)} g \times \theta\right)|_{k+2m+1/2}\alpha(z) &= \left(\left(\delta_k^{(m)} g\right)|_{k+2m}\alpha\right)(z) \times \left(\theta|_{1/2}\alpha\right)(z)\\ &=\delta_k^{(m)}\left(g|_k\alpha\right)(z) \times \left(\theta|_{1/2}\alpha\right)(z)\\ &=\sum_{j=0}^{m} d_j y^{-j}(g|_k\alpha)^{(j)}(z) \times (\theta|_{1/2}\alpha)(z)\end{aligned}$$

for some constants d_j. Since $(g|_k\alpha)^{(j)}$ and $\theta|_{1/2}\alpha$ are all holomorphic at ∞, so condition (2) of Definition 1.1 is verified for $\delta_k^{(m)} g(z) \times \theta(z)$. It is also clear that $\delta_k^{(m)} g(z) \times \theta(z)$ satisfies condition (3) because $g^{(j)}(z) = 4^j f^{(j)}(4z)$. Thus $\delta_k^{(m)} g(z) \times \theta(z) \in \widehat{M}^+_{k+2m+1/2}(4)$.

Finally, using Proposition 2.3, we can write

$$[\delta_k^{(n)} g(z), \theta(z)]_e = \sum_{j=0}^{e} a_j(e)\delta_{k+2n+2j+1/2}^{(e-j)}\left(\delta_k^{(n+j)} g(z) \times \theta(z)\right).$$

Since $\delta_{k+1/2}^{(m)} : \widehat{M}^+_{k+1/2} \to \widehat{M}^+_{k+2m+1/2}$ for $m \geq 0$, we conclude $[\delta_k^{(n)} g(z), \theta(z)]_e \in \widehat{M}^+_{k+2n+2e+1/2}$. □

3. Hecke equivariance of Shimura lifts

The goal of this section is two-folded: we first extend the definition of Hecke operators to nearly holomorphic modular forms, and then show that the extended Shimura lift $\mathcal{S}_D$ (1.6) is Hecke equivariant. The theory of Hecke operators on the holomorphic $M_k(1)$ is well known, so we will focus on Hecke operators on $M_{k+1/2}(4)$ and $\widehat{M}^+_{k+1/2}(4)$, following closely the exposition of [12]. Let G be the group of ordered pairs $(\alpha, \phi(z))$, where $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi(z)$ is a holomorphic function on $\mathbb{H}$ such that $|\phi(z)| = (\det\alpha)^{-1/4}|cz+d|^{1/2}$, with group law defined by

$$(\alpha, \phi(z))(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)).$$

The weight $k+1/2$ double slash operator of $\xi = (\alpha, \phi(z)) \in G$ on f is given by

$$\left(f||_{k+1/2}\xi\right)(z) := \phi(z)^{-2k-1} f(\alpha z) = \phi(z)^{-2k-1} f(\xi z).$$

There is a natural embedding of $\Gamma_0(4)$ into G given by

$$L : \alpha \mapsto \alpha^* = (\alpha, j(\alpha, z)),$$

where $j(\alpha, z) = \left(\frac{c}{d}\right)\varepsilon_d(cz+d)^{1/2}$ if $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It is the inverse of the projection mapping

$$P(\xi) = P((\alpha, \phi(z))) = \alpha.$$

Note that $M_{k+1/2}(4)$ consists of holomorphic functions f on $\mathbb{H}$ which satisfy $f||_{k+1/2}\alpha^* = f$ for every $\alpha \in \Gamma_0(4)$ and which are holomorphic at the cusps. We still denote the image of $\Gamma_0(4)$ in G as $\Gamma_0(4)$.

For a prime p, let $\alpha = \begin{bmatrix} 1 & 0 \\ 0 & p^2 \end{bmatrix}$ and $\xi = (\alpha, p^{1/2})$. Then the Hecke operator $T_{p^2,k+1/2}$ on $\widehat{M}_{k+1/2}(4)$ is given by

$$T_{p^2,k+1/2}(f) := p^{k-3/2} \sum_{\nu} f||_{k+1/2}\xi_\nu,$$

where ξ_ν are right coset representatives of $\Gamma_0(4)$ in $\Gamma_0(4)\xi\Gamma_0(4)$. It is more convenient to rewrite this definition in terms of single slash operators (1.2). We recall the following result of Shimura [12, p. 451] on an explicit set of right coset representatives.

Lemma 3.1 ([12])**.** A set of coset representatives of $\Gamma_0(4)\backslash\Gamma_0(4)\xi\Gamma_0(4)$ is given by

$$\alpha_b^* := (\alpha_b, p^{1/2}) = \left(\begin{bmatrix} 1 & b \\ 0 & p^2 \end{bmatrix}, p^{1/2} \right), \qquad 0 \le b < p^2,$$

$$\beta_h^* := (\beta_h, \varepsilon_p^{-1}\left(\frac{-h}{p}\right)) = \left(\begin{bmatrix} p & h \\ 0 & p \end{bmatrix}, \varepsilon_p^{-1}\left(\frac{-h}{p}\right) \right), \qquad 0 < h < p,$$

$$\sigma^* := (\sigma, p^{-1/2}) = \left(\begin{bmatrix} p^2 & 0 \\ 0 & 1 \end{bmatrix}, p^{-1/2} \right).$$

Thus, if $f(z) \in \widehat{M}_{k+1/2}(4)$, then [12, p. 451] gives

$$\begin{aligned}
&T_{p^2,k+1/2}(f) \\
=&p^{k-3/2}\left(\sum_b f||_{k+1/2}\alpha_b^* + \sum_h f||_{k+1/2}\beta_h^* + f||_{k+1/2}\sigma^* \right) \\
=&p^{-2}\sum_{b=0}^{p^2-1} f\left(\frac{z+b}{p^2}\right) + p^{k-3/2}\sum_{h=1}^{p-1}\varepsilon_p^{2k+1}\left(\frac{-h}{p}\right) f\left(\frac{pz+h}{p}\right) + p^{2k-1}f(p^2z) \\
=&p^{-2}\sum_b p^{k+1/2} f|_{k+1/2}p^{-1}\alpha_b + p^{k-3/2}\sum_{h=1}^{p-1}\varepsilon_p^{2k+1}\left(\frac{-h}{p}\right) f|_{k+1/2}p^{-1}\beta_h + p^{2k-1}p^{-k-1/2}f|_{k+1/2}p^{-1}\sigma \\
=&p^{k-3/2}\sum_b f|_{k+1/2}p^{-1}\alpha_b + p^{k-3/2}\sum_{h=1}^{p-1}\varepsilon_p^{2k+1}\left(\frac{-h}{p}\right) f|_{k+1/2}p^{-1}\beta_h + p^{k-3/2}f|_{k+1/2}p^{-1}\sigma. \qquad (3.1)
\end{aligned}$$

When $f(z) = \sum_{n\ge 0} a_f(n)q^n \in M_{k+1/2}(4)$, (3.1) can be further simplified as [12, Theorem 1.7]

$$T_{p^2,k+1/2}(f) = \sum_{n\ge0} a_f(p^2n)q^n + \sum_{n\ge0}\left(\frac{(-1)^k n}{p}\right)p^{k-1}a_f(n)q^n + \sum_{n\ge0} p^{2k-1}a_f\left(\frac{n}{p^2}\right)q^n. \quad (3.2)$$

Since the single slash operator commutes with $\delta_{k+1/2}^{(n)}$, we have from (3.1) that for $f \in \widehat{M}_{k+1/2}(4)$,

$$\delta_{k+1/2}^{(n)}\left(T_{p^2,k+1/2}(f)\right)$$

$$
\begin{aligned}
=&\delta_{k+1/2}^{(n)}\left(p^{k-3/2}f|_{k+1/2}p^{-1}\alpha_b+p^{k-3/2}\sum_{h=1}^{p-1}\varepsilon_p^{2k+1}\left(\frac{-h}{p}\right)f|_{k+1/2}p^{-1}\beta_h+p^{k-3/2}f|_{k+1/2}p^{-1}\sigma\right)\\
=&p^{k-3/2}(\delta_{k+1/2}^{(n)}f)|_{k+2n+1/2}p^{-1}\alpha_b+p^{k-3/2}\sum_{h=1}^{p-1}\varepsilon_p^{2k+1}\left(\frac{-h}{p}\right)(\delta_{k+1/2}^{(n)}f)|_{k+2n+1/2}p^{-1}\beta_h\\
&+p^{k-3/2}\left(\delta_{k+1/2}^{(n)}f\right)|_{k+2n+1/2}p^{-1}\sigma\\
=&p^{-2n}T_{p^2,k+2n+1/2}\left(\delta_{k+1/2}^{(n)}(f)\right). \qquad (3.3)
\end{aligned}
$$

On the other hand, for modular forms of integral weight, the same argument (or [2, Proposition 2.4]) gives us the following relation on Hecke operators for $g\in M_{2k}(1)$:

$$
\delta_{2k}^{(n)}\left(T_{p,2k}(g)\right)=p^{-n}T_{p,2k+2n}\left(\delta_{2k}^{(n)}(g)\right). \qquad (3.4)
$$

We also need to define Hecke operators on the subspace $\widehat{M}_{k+1/2}^{+}(4)$. If p is an odd prime, then the Hecke operator $T^{+}_{p^2,k+1/2}$ on $\widehat{M}_{k+1/2}^{+}(4)$ is exactly the restriction of the above $T_{p^2,k+1/2}$ to $\widehat{M}_{k+1/2}^{+}(4)$; see [7, Section 2.3] for the claim on $M_{k+1/2}^{+}(4)$ and their equality on $\widehat{M}_{k+1/2}^{+}(4)$ is due to (3.3). In this case, combining the commutativity established in (3.3) and (3.4) we obtain the following equivariance of Hecke operators for the Shimura lift $\mathcal{S}_D$ on $\widehat{M}_{k+1/2}^{+}(4)$.

Proposition 3.2. Let $f\in M_{k+1/2}^{+}(4)$ and D a fundamental discriminant such that $(-1)^kD>0$. Then for every prime p and $n\geq 0$ we have

$$
T_{p,2k+4n}\left(\mathcal{S}_D\left(\delta_{k+1/2}^{(n)}f\right)\right)=\mathcal{S}_D\left(T^{+}_{p^2,k+2n+1/2}\left(\delta_{k+1/2}^{(n)}f\right)\right).
$$

Consequently, for $g\in\widehat{M}_{k+1/2}^{+}$ and any prime p we have the following Hecke equivariance

$$
T_{p,2k}\left(\mathcal{S}_D(g)\right)=\mathcal{S}_D\left(T^{+}_{p^2,k+1/2}(g)\right).
$$

Proof. When p is odd, as discussed above, we have $T^{+}_{p^2,k+1/2}(f)=T_{p^2,k+1/2}(f)$. Thus

$$
\begin{aligned}
T_{p,2k+4n}\left(\mathcal{S}_D\left(\delta_{k+1/2}^{(n)}f\right)\right)&=|D|^nT_{p,2k+4n}\left(\delta_{2k}^{(2n)}\left(\mathcal{S}_D(f)\right)\right) && \text{by (1.6)}\\
&=p^{2n}|D|^n\delta_{2k}^{(2n)}\left(T_{p,2k}\left(\mathcal{S}_D(f)\right)\right) && \text{by (3.4)}\\
&=p^{2n}|D|^n\delta_{2k}^{(2n)}\left(\mathcal{S}_D\left(T^{+}_{p^2,k+1/2}f\right)\right) && \text{by holomorphic equivariance}\\
&=p^{2n}\mathcal{S}_D\left(\delta_{k+1/2}^{(n)}\left(T^{+}_{p^2,k+1/2}f\right)\right) && \text{by (1.6)}\\
&=\mathcal{S}_D\left(T^{+}_{p^2,k+2n+1/2}(\delta_{k+1/2}^{(n)}f)\right), && \text{by (3.3)}
\end{aligned}
$$

as desired.

When $p=2$, the same argument also applies by the discussion below. □

At last, for completeness, we consider the case of $T^+_{4,k+1/2}$, that is when $p = 2$. Recall that Kohnen [7, p. 250] defined $T^+_{4,k+1/2}$ on $f \in M^+_{k+1/2}(4)$ as

$$T^+_{4,k+1/2}(f) := \sum_{(-1)^k n \equiv 0,1 \pmod 4} \left(a_f(4n) + \left(\frac{(-1)^k n}{2}\right) 2^{k-1} a_f(n) + 2^{2k-1} a_f\left(\frac{n}{4}\right) \right) q^n. \quad (3.5)$$

We again need to rewrite this Hecke operator in terms of slash operators. Since $f \in M^+_{k+1/2}$ and thus $a_f(n) = 0$ for $(-1)^k n \equiv 2, 3 \pmod 4$, we only need to rewrite the q-series in (3.5) involving $a_f(4n)$ because the other two q-series remain the same as those in $T_{4,k+1/2}(f)$ by (3.2). This has been done by Kohnen [7, (1)]:

$$\sum_{(-1)^k n \equiv 0,1 \pmod 4} a_f(4n) q^n = s_1(f) + s_3(f) + s_4(f).$$

Here for $v = 1, 2, 3$,

$$s_v(f) = 2^{k-2+1/2} \left((f||_{k+1/2} W_4)||_{k+1/2} \alpha_v^* \right) ||_{k+1/2} W_4, \quad s_4(f) = \left(\frac{2}{2k+1}\right) 2^{k-1} f||_{k+1/2} W_4,$$

where $\alpha_v^* := (\alpha_v, 2^{1/2}) = \left(\begin{bmatrix} 1 & v \\ 0 & 4 \end{bmatrix}, 2^{1/2} \right)$. Note that for $\gamma = \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix}$,

$$\left(f||_{k+1/2} W_4\right)(z) := (-2iz)^{-k-1/2} f\left(-\frac{1}{4z}\right) = (-i)^{-k-1/2} \left(f|_{k+1/2}\gamma\right)(z),$$

see Xue [16, (3.3)]. At the same time, (3.1) shows that $g||_{k+1/2}\alpha_v^*$ is a multiple of a single slash operator. In other words, the first term $\sum_{(-1)^k n \equiv 0,1 \pmod 4} a_f(4n) q^n$ in (3.5) can also be written as a linear combination of single slash operators. Hence, the argument of (3.3) still works for $T^+_{4,k+1/2}$ and thus the Hecke equivariance claim in Proposition 3.2 continues to hold for $p = 2$.

4. Selberg identity for $\mathcal{S}_1$

This section is devoted to the proof of Theorem 1.2, one of our main results. We begin by establishing the following hypergeometric sum identity.

Lemma 4.1. Let e, m be nonnegative integers with $m \leq 2e$ and $k > 0$ a real number. Then,

$$\sum_{j=0}^{m} (-4)^j \binom{k+e-1}{e-j} \binom{k+e+j-3/2}{j} \binom{2e-2j}{m-j} = (-1)^m \frac{\binom{k+e-1}{e}\binom{k+2e-1}{2e-m}\binom{k+2e-1}{m}}{\binom{k+2e-1}{2e}}.$$

Proof. We recognize this sum as a terminating hypergeometric series. A computation shows that

$$(-4)^j \binom{k+e-1}{e-j} \binom{k+e+j-3/2}{j} \binom{2e-2j}{m-j} = \binom{2e}{m} \binom{k+e-1}{e} \frac{(-2e+m)_j (k+e-\frac{1}{2})_j (-m)_j}{j!(k)_j(-e+\frac{1}{2})_j},$$

and we note

$${}_3F_2\left(\begin{matrix} -2e+m, k+e-\frac{1}{2}, -m \\ k, -e+\frac{1}{2} \end{matrix} ; 1 \right) = \sum_{j=0}^{m} \frac{(-2e+m)_j (k+e-\frac{1}{2})_j (-m)_j}{j!(k)_j(-e+\frac{1}{2})_j}.$$

One checks that $-e+1/2 = 1+(-2e+m)+(k+e-1/2)-k-m$, and m is a nonnegative integer, so we may apply Saalschütz's theorem [1, p. 9] to obtain

$$ {}_3F_2\left(\begin{matrix} -2e+m, k+e-\frac{1}{2}, -m \\ k, -e+\frac{1}{2} \end{matrix} ;1\right) = \frac{(k+2e-m)_m(\frac{1}{2}-e)_m}{(k)_m(e-m+\frac{1}{2})_m} = (-1)^m\frac{(k+2e-m)_m}{(k)_m}. $$

This gives us

$$ \begin{aligned} \sum_{j=0}^{m}(-4)^j\binom{k+e-1}{e-j}\binom{k+e+j-3/2}{j}\binom{2e-2j}{m-j} &= (-1)^m\binom{2e}{m}\binom{k+e-1}{e}\frac{(k+2e-m)_m}{(k)_m} \\ &= (-1)^m\frac{\binom{k+e-1}{e}\binom{k+2e-1}{2e-m}\binom{k+2e-1}{m}}{\binom{k+2e-1}{2e}}, \end{aligned} $$

so the claimed identity holds. □

Now, we prove the $e=0$ case of Theorem 1.2.

Proposition 4.2. Let $f \in M_k(1)$ be a normalized Hecke eigenform of weight $k \geq 4$. Furthermore, let $n \geq 0$ be an integer. Then,

$$ \mathcal{S}_1\left(\delta_k^{(n)}(f(4z)) \times \theta(z)\right) = 4^n\left(\delta_k^{(n)}f(z)\right)^2. \tag{4.1} $$

Proof. We proceed by induction on n. In the $n=0$ case, this is the known Selberg identity (1.4). Now, let $n \geq 1$ and suppose that (4.1) holds for $0 \leq j < n$. By Proposition 2.3, we have

$$ \begin{aligned} &(-1)^n\binom{k+2n-3/2}{n}\delta_k^{(n)}(f(4z)) \times \theta(z)) \\ =&[f(4z),\theta(z)]_n - \sum_{j=0}^{n-1}a_j(n)\delta_{k+2j+1/2}^{(n-j)}(\delta_k^{(j)}(f(4z)) \times \theta(z)), \end{aligned} \tag{4.2} $$

where for $0 \leq j < n$

$$ a_j(n) = (-1)^j\binom{k+n-1}{n-j}\binom{k+n+j-3/2}{j}. $$

Now, we take $\mathcal{S}_1$ of both sides of (4.2). Using (1.5) and (1.6) as well as our induction hypothesis on the right, we get

$$ \frac{\binom{k+n-1}{n}}{\binom{k+2n-1}{2n}}[f(z),f(z)]_{2n} - \sum_{j=0}^{n-1}4^j a_j(n)\delta_{2k+4j}^{(2n-2j)}\left(\left(\delta_k^{(j)}f(z)\right)^2\right). \tag{4.3} $$

So by (4.2) and (4.3),

$$ \mathcal{S}_1\left(\delta_k^{(n)}\left(f(4z)\right) \times \theta(z)\right) = (-1)^n\frac{A}{\binom{k+2n-3/2}{n}}, \tag{4.4} $$

where A is exactly (4.3).

Now, we examine A by finding the coefficient of $\delta_k^{(m)} f(z) \times \delta_k^{(2n-m)} f(z)$ in it for $0 \le m \le n$ through purely algebraic manipulations. Applying the Leibniz rule for $\delta_{2k+4j}^{(2n-2j)}$, we obtain

$$\delta_{2k+4j}^{(2n-2j)} \left(\left(\delta_k^{(j)} f(z) \right)^2 \right) = \sum_{d=0}^{2n-2j} \binom{2n-2j}{d} \delta_k^{(j+d)} f(z) \times \delta_k^{(2n-j-d)} f(z).$$

Using this and Proposition 2.2, we obtain that in the case $0 \le m < n$, the coefficient is

$$2 \left((-1)^m \frac{\binom{k+n-1}{n}}{\binom{k+2n-1}{2n}} \binom{k+2n-1}{2n-m} \binom{k+2n-1}{m} \right.$$
$$\left. - \sum_{j=0}^{m} (-4)^j \binom{k+n-1}{n-j} \binom{k+n+j-3/2}{j} \binom{2n-2j}{m-j} \right),$$

which is seen to be zero by applying Lemma 4.1 with $e = n$. In the case $m = n$, we similarly find the coefficient of $\delta_k^{(n)} f(z) \times \delta_k^{(n)} f(z)$ in A is

$$(-1)^n \frac{\binom{k+n-1}{n}}{\binom{k+2n-1}{2n}} \binom{k+2n-1}{n}^2 - \sum_{j=0}^{n-1} (-4)^j \binom{k+n-1}{n-j} \binom{k+n+j-3/2}{j} \binom{2n-2j}{n-j}.$$

This sum is equal to

$$\sum_{j=0}^{n} (-4)^j \binom{k+n-1}{n-j} \binom{k+n+j-3/2}{j} \binom{2n-2j}{n-j} - (-4)^n \binom{k+2n-3/2}{n},$$

and applying Lemma 4.1 with $e = m = n$ gives

$$\sum_{j=0}^{n} (-4)^j \binom{k+n-1}{n-j} \binom{k+n+j-3/2}{j} \binom{2n-2j}{n-j} = (-1)^n \frac{\binom{k+n-1}{n}}{\binom{k+2n-1}{2n}} \binom{k+2n-1}{n}^2.$$

So we get

$$A = (-4)^n \binom{k+2n-3/2}{n} \left(\delta_k^{(n)} f(z) \right)^2,$$

and thus by (4.4) we have

$$\mathcal{S}_1 \left(\delta_k^{(n)} (f(4z)) \times \theta(z) \right) = (-1)^n \frac{(-4)^n \binom{k+2n-3/2}{n} \left(\delta_k^{(n)} f(z) \right)^2}{\binom{k+2n-3/2}{n}} = 4^n \left(\delta_k^{(n)} f(z) \right)^2,$$

as desired. □

Now, we can prove Theorem 1.2 in full generality.

Theorem 1.2. Let $f \in M_k(1)$ be a normalized Hecke eigenform of weight $k \ge 4$. Furthermore, let $n, e \ge 0$ be integers. Then,

$$\mathcal{S}_1 \left([\delta_k^{(n)} (f(4z)), \theta(z)]_e \right) = 4^n \frac{\binom{k+e+2n-1}{e}}{\binom{k+2e+2n-1}{2e}} [\delta_k^{(n)} f(z), \delta_k^{(n)} f(z)]_{2e}.$$

Proof. The $e=0$ case is done in Proposition 4.2, so suppose $e \geq 1$. Using Proposition 2.3, we get

$$[\delta_k^{(n)}(f(4z)), \theta(z)]_e = \sum_{j=0}^{e} a_j(e)\delta_{k+2n+1/2+2j}^{(e-j)}(\delta_k^{(n+j)}(f(4z)) \times \theta(z)),$$

where for $0 \leq j \leq e$,

$$a_j(e) = (-1)^j \binom{k+2n+e-1}{e-j}\binom{k+2n+e+j-3/2}{j}.$$

Using the Leibniz rule and applying Proposition 4.2 together with (1.6) using $D=1$, we get that

$$\begin{aligned}
\mathcal{S}_1\left([\delta_k^{(n)}(f(4z)), \theta(z)]_e\right) &= \sum_{j=0}^{e} a_j(e)\delta_{2k+4n+4j}^{(2e-2j)}\mathcal{S}_1\left(\delta_k^{(n+j)}(f(4z)) \times \theta(z)\right) \\
&= 4^n \sum_{j=0}^{e} 4^j a_j(e)\delta_{2k+4n+4j}^{(2e-2j)}\left(\left(\delta_k^{(n+j)}f(z)\right)^2\right) \\
&= 4^n \sum_{j=0}^{e} 4^j a_j(e) \sum_{d=0}^{2e-2j}\binom{2e-2j}{d}\delta_k^{(n+j+d)}f(z) \times \delta_k^{(n+2e-j-d)}f(z).
\end{aligned}$$

Now, we find the coefficient of $\delta_k^{(n+m)}f(z) \times \delta_k^{(n+2e-m)}f(z)$ with $0 \leq m \leq e$. In the case $0 \leq m < e$, the coefficient is

$$2 \times 4^n \sum_{j=0}^{m}(-4)^j\binom{k+2n+e-1}{e-j}\binom{k+2n+e+j-3/2}{j}\binom{2e-2j}{m-j},$$

which by Lemma 4.1 (applied with $k+2n$ instead of k) is equal to

$$2 \times 4^n \frac{\binom{k+2n+e-1}{e}}{\binom{k+2n+2e-1}{2e}} \times (-1)^m \binom{k+2n+2e-1}{2e-m}\binom{k+2n+2e-1}{m}.$$

Meanwhile, in the case $m=e$, the coefficient of $\left(\delta_k^{(n+e)}f(z)\right)^2$ is

$$4^n \sum_{j=0}^{e}(-4)^j\binom{k+2n+e-1}{e-j}\binom{k+2n+1/2+e+j-2}{j}\binom{2e-2j}{e-j},$$

which by similarly applying Lemma 4.1 in the case $m=e$ is seen to be equal to

$$4^n \frac{\binom{k+2n+e+1}{e}}{\binom{k+2n+2e+1}{2e}} \times (-1)^e \binom{k+2n+2e-1}{e}^2.$$

Since

$$\begin{aligned}
&2\sum_{j=0}^{e-1}(-1)^j\binom{k+2n+2e-1}{2e-j}\binom{k+2n+2e-1}{j}\delta_k^{(n+j)}f(z) \times \delta_k^{(n+2e-j)}f(z) \\
&= \sum_{\substack{0 \leq j \leq 2e \\ j \neq e}}(-1)^j\binom{k+2n+2e-1}{2e-j}\binom{k+2n+2e-1}{j}\delta_{k+2n}^{(j)}\left(\delta_k^{(n)}f(z)\right) \times \delta_{k+2n}^{(2e-j)}\left(\delta_k^{(n)}f(z)\right),
\end{aligned}$$

we have that $\mathcal{S}_1\left([\delta_k^{(n)}(f(4z)),\theta(z)]_e\right) =$

$$\frac{4^n\binom{k+2n+e+1}{e}}{\binom{k+2n+2e+1}{2e}}\sum_{j=0}^{2e}(-1)^j\binom{k+2n+2e-1}{2e-j}\binom{k+2n+2e-1}{j}\delta_{k+2n}^{(j)}\left(\delta_k^{(n)}f(z)\right)\delta_{k+2n}^{(2e-j)}\left(\delta_k^{(n)}f(z)\right).$$

We easily recognize the last sum without the factor in front as $[\delta_k^{(n)}f(z),\delta_k^{(n)}f(z)]_{2e}$ by Proposition 2.2 and thus conclude the proof. $\square$

5. Selberg identity for $\mathcal{S}_D$

We want to get a result similar to Theorem 1.2 for the D-th Shimura lift $\mathcal{S}_D$ for a general D. To achieve this we need first to introduce some notation.

Let $k \geq 4$ and $e \geq 0$ be integers with $\ell = k+2e$ and let D be an odd fundamental discriminant such that $(-1)^\ell D > 0$. The Eisenstein series $G_{k,D}$ and $G_{k,4D}$ are given by [6, p. 185]

$$G_{k,D}(z) := \frac{L_D(1-k)}{2} + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}\right)q^n \in M_k\left(|D|,\left(\frac{D}{\cdot}\right)\right),$$
$$G_{k,4D}(z) := G_{k,D}(4z) - 2^{-k}\left(\frac{D}{2}\right)G_{k,D}(2z) \in M_k\left(4|D|,\left(\frac{D}{\cdot}\right)\right),$$

where $L_D(s) = \sum_{n\geq 1}\left(\frac{D}{n}\right)n^{-s}$. Recall two functions from [5]:

$$\mathcal{F}_{D,k,e}(z) := \mathrm{Tr}_1^D([G_{k,D}(z), G_{k,D}(z)]_{2e}) \in S_{2\ell}(1),$$
$$\mathcal{G}_{D,k,e}(z) := \mathrm{Tr}_4^{4D}\left([G_{k,D}(4z),\theta(|D|z)]_e\right).$$

This definition of $\mathcal{G}_{D,k,e}$ is equivalent to the one presented in [5, (1.12)] by [5, Proposition 3.9]. Here Tr_M^N for $M \mid N$ is the trace map from modular forms of level N to level M. We now define two corresponding functions in the nearly holomorphic setting:

$$\mathcal{F}_{D,k,e}^{(n)}(z) := \mathrm{Tr}_1^D\left([\delta_k^{(n)}G_{k,D}(z), \delta_k^{(n)}G_{k,D}(z)]_{2e}\right),$$
$$\mathcal{G}_{D,k,e}^{(n)}(z) := \mathrm{Tr}_4^{4D}\left([\delta_k^{(n)}\left(G_{k,D}(4z)\right),\theta(|D|z)]_e\right).$$

Lemma 5.1. We have that

$$\mathcal{G}_{D,k,e}^{(n)}(z) \in \widehat{M}^+_{k+2n+2e+1/2}(4).$$

Proof. By [5, Lemma 3.1] or [6, p. 196], as a set of representatives for $\Gamma_0(4D)\backslash\Gamma_0(4)$ we can take

$$\gamma_{D_1,\mu} = \begin{bmatrix}1 & 0\\ 4|D_1| & 1\end{bmatrix}\begin{bmatrix}1 & \mu\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & \mu\\ 4|D_1| & 4\mu|D_1|+1\end{bmatrix}, \quad D = D_1D_2,\ \mu \mod D_2,$$

where D_1, D_2 are fundamental discriminants. To ease notation we simply write this set of representatives as $\{\gamma_j\}$. It is straightforward to verify that for each γ_j and integer $k \geq 1$

$$f||_{k+1/2}\gamma_j = f|_{k+1/2}\gamma_j,$$

a fact which was used implicitly in both [6] and [5]. This implies that for $g \in \widehat{M}_{k+2n+2e+1/2}(4)$

$$\mathrm{Tr}_4^{4D}(g) := \sum_j g||_{k+2n+2e+1/2}\gamma_j = \sum_j g|_{k+2n+2e+1/2}\gamma_j.$$

Since Maass-Shimura operators commute with both the slash and double slash operators, this makes it clear that they commute with the trace map Tr_4^{4D} as well.

To prove our desired result, we will first show that $\mathrm{Tr}_4^{4D}(\delta_k^{(n)}G_{k,D}(4z) \times \theta(|D|z))$ is in the plus space by induction on n. This is known when $n = 0$ by [6], so let $n \geq 1$ and assume that $\mathrm{Tr}_4^{4D}(\delta_k^{(j)}G_{k,D}(4z) \times \theta(|D|z))$ is in the plus space for $0 \leq j < n$. Then, by Proposition 2.3, linearity of the trace, and commutativity of the trace map and Maass-Shimura operators, we have

$$\begin{aligned}&a_n(n)\,\mathrm{Tr}_4^{4D}\left(\delta_k^{(n)}G_{k,D}(4z) \times \theta(|D|z)\right)\\ &= \mathrm{Tr}_4^{4D}\left([G_{k,D}(4z), \theta(|D|z)]_n\right) - \sum_{j=0}^{n-1} a_j(n)\delta_{k+1/2+2j}^{(n-j)}\left(\mathrm{Tr}_4^{4D}\left(\delta_k^{(j)}G_{k,D}(4z) \times \theta(|D|z)\right)\right).\end{aligned}$$

We know the first term is in the plus space by [5], and that $\mathrm{Tr}_4^{4D}(\delta_k^{(j)}G_{k,D}(4z) \times \theta(|D|z)) \in \widehat{M}^+_{k+2j+1/2}(4)$ by the induction hypothesis. Since the Maass-Shimura operator preserves the plus space, this shows that $\mathrm{Tr}_4^{4D}(\delta_k^{(n)}G_{k,D}(4z) \times \theta(|D|z)) \in \widehat{M}^+_{k+2n+1/2}(4)$.

Now, using Proposition 2.3, we can write

$$\begin{aligned}\mathrm{Tr}_4^{4D}([\delta_k^{(n)}(G_{k,D}(4z)), \theta(|D|z)]_e) &= \mathrm{Tr}_4^{4D}\left(\sum_{j=0}^{e} a_j(e)\delta_{k+2n+2j+1/2}\left(\delta_k^{(n+j)}(G_{k,D}(4z)) \times \theta(|D|z)\right)\right)\\ &= \sum_{j=0}^{e} a_j(e)\delta_{k+2n+2j+1/2}\left(\mathrm{Tr}_4^{4D}\left(\delta_k^{(n+j)}(G_{k,D}(4z)) \times \theta(|D|z)\right)\right).\end{aligned}$$

By the argument above, each $\mathrm{Tr}_4^{4D}(\delta_k^{(n+j)}(G_{k,D}(4z)) \times \theta(|D|z))$ is in the plus space, from which the conclusion follows. □

The next theorem gives the desired extension of Theorem 1.2 to $\mathcal{S}_D$.

Theorem 5.2. Let $k \geq 4$, $n, e \geq 0$ be integers, and D an odd fundamental discriminant such that $(-1)^k D > 0$. Then we have

$$\mathcal{S}_D\left(\mathcal{G}_{D,k,e}^{(n)}(z)\right) = 4^n|D|^{n+e}\frac{\binom{k+e+2n-1}{e}}{\binom{k+2e+2n-1}{2e}}\mathcal{F}_{D,k,e}^{(n)}(z).$$

Similar to the proof of Theorem 1.2, we will first prove the $e = 0$ case.

Proposition 5.3. With the assumptions of Theorem 5.2, we have

$$\mathcal{S}_D\left(\mathrm{Tr}_4^{4D}\left(\delta_k^{(n)}\left(G_{k,D}(4z)\right)\times\theta(|D|z)\right)\right)=4^n|D|^n\,\mathrm{Tr}_1^D\left(\left(\delta_k^{(n)}G_{k,D}(z)\right)^2\right).$$

Proof. We proceed by induction on n. In the $n=0$ case, this is known; see [5, Theorem 1.1], or Kohnen-Zagier [6, Proposition 3]. Now, let $n\geq 1$ and assume that the identity holds when n is replaced with j for all $0\leq j<n$. By Proposition 2.3 and linearity of the trace map, we have

$$\begin{aligned}&a_n(n)\,\mathrm{Tr}_4^{4D}\left(\delta_k^{(n)}(G_{k,D}(4z))\times\theta(|D|z)\right)\\&=\mathrm{Tr}_4^{4D}\left([G_{k,D}(4z),\theta(|D|z)]_n\right)-\sum_{j=0}^{n-1}a_j(n)\,\mathrm{Tr}_4^{4D}\left(\delta_{k+1/2+2j}^{(n-j)}(\delta_k^{(j)}(G_{k,D}(4z))\times\theta(|D|z))\right),\end{aligned}\tag{5.1}$$

where for $0\leq j\leq n$

$$a_j(n)=(-1)^j\binom{k+n-1}{n-j}\binom{k+n+j-3/2}{j}.$$

Now we take $\mathcal{S}_D$ of both sides of (5.1). On the left, we have

$$(-1)^n\binom{k+2n-3/2}{n}\mathcal{S}_D\left(\mathrm{Tr}_4^{4D}\left(\delta_k^{(n)}(G_{k,D}(4z))\times\theta(|D|z)\right)\right).$$

Meanwhile, on the right of (5.1) the first term is

$$\mathcal{S}_D\left(\mathrm{Tr}_4^{4D}\left([G_{k,D}(4z),\theta(|D|z)]_n\right)\right)=|D|^n\frac{\binom{k+n-1}{n}}{\binom{k+2n-1}{2n}}\,\mathrm{Tr}_1^D\left([G_{k,D}(z),G_{k,D}(z)]_{2n}\right)$$

by [5, Theorem 1.1], while the second term is $\mathcal{S}_D$ of a sum we handle as follows. By applying (1.6), the induction hypothesis, linearity of the trace map, and the fact that Maass-Shimura operators and trace map commute (shown in the proof of Lemma 5.1) we obtain

$$\begin{aligned}&\mathcal{S}_D\left(\sum_{j=0}^{n-1}a_j(n)\,\mathrm{Tr}_4^{4D}\left(\delta_{k+1/2+2j}^{(n-j)}\left(\delta_k^{(j)}(G_{k,D}(4z))\times\theta(|D|z)\right)\right)\right)\\&=\sum_{j=0}^{n-1}a_j(n)|D|^{n-j}\delta_{2k+4j}^{(2n-2j)}\left(\mathcal{S}_D\left(\mathrm{Tr}_4^{4D}\left(\delta_k^{(j)}(G_{k,D}(4z))\times\theta(|D|z)\right)\right)\right)\\&=|D|^n\,\mathrm{Tr}_1^D\left(\sum_{j=0}^{n-1}4^ja_j(n)\delta_{2k+4j}^{(2n-2j)}\left(\left(\delta_k^{(j)}G_{k,D}(z)\right)^2\right)\right).\end{aligned}$$

So after applying $\mathcal{S}_D$ to (5.1), we obtain

$$\mathcal{S}_D\left(\mathrm{Tr}_4^{4D}\left(\delta_k^{(n)}(G_{k,D}(4z))\times\theta(|D|z)\right)\right)=|D|^n\times(-1)^n\frac{\mathrm{Tr}_1^D(B)}{\binom{k+2n-3/2}{n}},\tag{5.2}$$

where

$$B=\frac{\binom{k+n-1}{n}}{\binom{k+2n-1}{2n}}[G_{k,D}(z),G_{k,D}(z)]_{2n}-\sum_{j=0}^{n-1}4^ja_j(n)\delta_{2k+4j}^{(2n-2j)}\left(\left(\delta_k^{(j)}G_{k,D}(z)\right)^2\right).$$

Notice that B is the same as (4.3), but with $G_{k,D}$ instead of f. So, following the same algebraic manipulations done at the end of the proof of Proposition 4.2, we see

$$B = (-4)^n \binom{k+2n-3/2}{n} \left(\delta_k^{(n)} G_{k,D}(z)\right)^2.$$

Thus by (5.2)

$$\begin{aligned}\mathcal{S}_D\left(\mathrm{Tr}_4^{4D}\left(\delta_k^{(n)}(G_{k,D}(4z)) \times \theta(|D|z)\right)\right) &= (-1)^n |D|^n \frac{\mathrm{Tr}_1^D\left((-4)^n\binom{k+2n-3/2}{n}(\delta_k^{(n)}G_{k,D}(z))^2\right)}{\binom{k+2n-3/2}{n}} \\ &= 4^n |D|^n \,\mathrm{Tr}_1^D\left((\delta_k^{(n)}G_{k,D}(z))^2\right),\end{aligned}$$

as desired. □

Now, we handle the cases where $e \geq 1$ to conclude Theorem 5.2.

Proof of Theorem 5.2. Let $e \geq 1$, as we have already proven the $e = 0$ case. Using (2.5), we get

$$\mathrm{Tr}_4^{4D}\left(\left[\delta_k^{(n)}G_{k,D}(4z), \theta(|D|z)\right]_e\right) = \sum_{j=0}^{e} a_j(e)\,\mathrm{Tr}_4^{4D}\left(\delta_{k+2n+1/2+2j}^{(e-j)}\left(\delta_k^{(n+j)}G_{k,D}(z) \times \theta(|D|z)\right)\right),$$

where

$$a_j(e) = (-1)^j \binom{k+2n+e-1}{e-j}\binom{k+2n+\frac{1}{2}+e+j-2}{j}.$$

Using the commutativity of Maass-Shimura operators with the trace map, (1.6), and Proposition 5.3, we can apply $\mathcal{S}_D$ and write the right hand side as

$$\begin{aligned}&\mathcal{S}_D\left(\sum_{j=0}^{e} a_j(e)\,\mathrm{Tr}_4^{4D}(\delta_{k+2n+1/2+2j}^{(e-j)}(\delta_k^{(n+j)}G_{k,D}(z) \times \theta(|D|z)))\right) \\ =&4^n|D|^{n+e}\sum_{j=0}^{e} 4^j a_j(e)\delta_{2k+4j+4n}^{(2e-2j)}\left(\mathrm{Tr}_1^D\left((\delta_k^{(n+j)}G_{k,D}(z))^2\right)\right) \\ =&4^n|D|^{n+e}\,\mathrm{Tr}_1^D\left(\sum_{j=0}^{e} 4^j a_j(e)\delta_{2k+4j+4n}^{(2e-2j)}\left((\delta_k^{(n+j)}G_{k,D}(z))^2\right)\right).\end{aligned}$$

Following the same algebraic manipulations performed in the proof of Theorem 1.2, we have

$$\sum_{j=0}^{e} 4^j a_j(e)\delta_{2k+4j+2n}^{(2e-2j)}\left(\left(\delta_k^{(n+j)}G_{k,D}(z)\right)^2\right) = \frac{\binom{k+e+2n-1}{e}}{\binom{k+2e+2n-1}{2e}}[\delta_k^{(n)}G_{k,D}(z), \delta_k^{(n)}G_{k,D}(z)]_{2e},$$

from which it follows that $\mathcal{S}_D\left(\mathrm{Tr}_4^{4D}\left(\left[\delta_k^{(n)}G_{k,D}(4z), \theta(|D|z)\right]_e\right)\right) =$

$$4^n|D|^{e+n}\frac{\binom{k+2n+e+1}{e}}{\binom{k+2n+2e+1}{2e}}\,\mathrm{Tr}_1^D\left(\left[\delta_k^{(n)}G_{k,D}(z), \delta_k^{(n)}G_{k,D}(z)\right]_{2e}\right).$$

By recalling the definitions of $\mathcal{G}_{D,k,e}^{(n)}$ and $\mathcal{F}_{D,k,e}^{(n)}$, we see this is the desired result. □

6. Discussion

First, the notion of nearly holomorphic modular forms of half-integral weight can be generalized to congruence subgroups of $\Gamma_0(4)$ in an obvious way, as well as the theory of Hecke operators. Using (1.6), Shimura lifts can also be extended to these modular forms.

Waldspurger [15] has worded out the Shimura correspondence in the framework of automorphic representations. Roughly speaking, to each irreducible automorphic representation $\widetilde{\pi}$ of the metaplectic cover of SL_2 Waldspurger determined a representation π of GL_2, whose L-function is related to that of $\widetilde{\pi}$ in the way dictated by Shimura. Since the Maass-Shimura operators only affect the archimedean places and do not change the underlying automorphic representations $\widetilde{\pi}$ or π, it is expected that they are related naturally via Waldspurger. This is one of the motivations of our paper.

Our definition of the Shimura lift $\mathcal{S}_D$ in (1.6) is somehow artificial, although Proposition 3.2 shows that it satisfies the critical Hecke equivariance property. It is highly desirable to construct such correspondence in a more natural way, for example through the theta correspondence with a suitable kernel function. We note that Niwa [10] (and Shintani [14]) have constructed the Shimura correspondence (and its adjoint) through explicit theta correspondences. Recently, Shintani's construction has been generalized to the nearly holomorphic setting by Li and Zemel [9]. We hope to generalize Niwa's construction to the nearly holomorphic modular form setting and show it matches our definition. Also, we hope to study the relationship between our construction and the one in [9]. In particular, we want to investigate whether these two constructions are adjoint to each other, like the relationship between the constructions of Niwa and Shintani.

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