Notes on the Prime Number Theorem

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This is an adapted version of some notes I took on the prime number theorem for a directed reading program (DRP) on analytic number theory. Much (but not all) of these notes were primarily taken from *Complex Analysis* by Elias M. Stein and Rami Shakarchi. Additional notes are mainly on things my DRP mentor, Nicholas Backes, shared with me, as well as certain online reading such as from certain Wikipedia articles.

1 Preliminaries

Basic complex analysis is assumed. Some other helpful tools are included below.

1.1 Some useful facts from Fourier analysis

The Fourier transform of f is a function $\widehat{f}$ given by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,dx$$

and this transformation can be inverted (under certain conditions) to recover back f with

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i\xi x}\,d\xi.$$

We can think about the Fourier transform as a kind of projection onto a vector space with basis vectors $\{e_\xi\}$ using the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$$

(the overline indicates complex conjugate).

Definition 1.1 (Schwartz function)**.** *A C^∞ function f is said to be Schwartz if $x^n f^{(k)}(x) \to 0$ as $|x| \to \infty$ for all $k \geq 0$; in words, f and all of its derivatives decay faster than all polynomials grow.*

For example, e^{-x^2} is Schwartz (an important example of such a function–it will be relevant later on). A key fact about the space of Schwartz functions is the following.

Theorem 1.2. *Let $\mathcal{S}$ be the set of Schwartz functions (in this case, over $\mathbb{R}$). Then the Fourier transform is an automorphism on $\mathcal{S}$.*

As an example, consider the Fourier transform of $f(x) = e^{-x^2}$:

$$\begin{aligned}\widehat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-x^2} e^{-2\pi i \xi x}\, dx = \int_{-\infty}^{\infty} e^{-x^2-2\pi i\xi x}\, dx = \int_{-\infty}^{\infty} e^{-(x+\pi i \xi)^2 - \pi^2\xi^2}\, dx \\ &= e^{-\pi^2\xi^2} \int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi} e^{-\pi^2\xi^2}\end{aligned}$$

which is another Gaussian (and must indeed be Schwartz as well).

1.1.1 The Poisson summation formula

Consider a function $f \in \mathcal{S}$. As $\lim_{x\to\pm\infty} x^2 f(x) = 0$, we know that the sum

$$\sum_{n\in\mathbb{Z}} f(n)$$

converges. The Poisson summation formula provides a useful technique for evaluating such sums.

Theorem 1.3 (Poisson summation formula)**.** *For a Schwarz function f,*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n).$$

Proof. Consider the function

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

Clearly F is periodic with period 1, as

$$F(x+1) = \sum_{n\in\mathbb{Z}} f(x+n+1) = \sum_{m\in\mathbb{Z}} f(x+m)$$

where $m = n+1$ (some simple reindexing). Thus, we can find a Fourier series $\sum_{n\in\mathbb{Z}} a_n e^{2\pi i n x}$ for F, with Fourier coefficients

$$\begin{aligned}a_n &= \int_0^1 F(x) e^{-2\pi i n x}\, dx = \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n) e^{-2\pi i n x}\, dx \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i n x}\, dx = \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(t) e^{-2\pi i n (t-n)}\, dt \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(t) e^{-2\pi i n t}\, dt = \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t}\, dt = \widehat{f}(n)\end{aligned}$$

This means that

$$\sum_{n\in\mathbb{Z}} f(x+n) = F(x) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$$

Now set $x = 0$ and we are done. □

1.2 Mellin transform

The Mellin transform of a function f is given by

$$\mathcal{M}(f)(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

and the inverse Mellin transform of φ is given by

$$f(x) = \frac{1}{2\pi i} \int_{\mathrm{Re}(z)=c} x^{-s} \varphi(s)\, dx.$$

Notice that is is an integral taken over the line $x = c$ in the complex plane. We take this integral moving upwards along the line.

As an example, the Mellin transform of the function $f(x) = e^{-x}$ is

$$\mathcal{M}(f)(s) \int_0^\infty x^{s-1} e^{-x}\, dx = \Gamma(s)$$

where Γ is the gamma function, which will be of use later.

Though not strictly required here, the Mellin transform can provide a useful, alternative way of looking at some of the results that will be presented.

2 The gamma function

The gamma function is defined for $s > 0$ by

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx.$$

This extends to an analytic function in the half-plane $\mathrm{Re}(s) > 0$ and is given by the same formula for such complex s. We can observe by integration by parts that $\Gamma(s+1) = s\Gamma(s)$, and as a consequence it follows that $\Gamma(n+1) = n!$ for nonnegative integers n (since $\Gamma(1) = 1$).

We can actually extend this further, and take an analytic continuation of Γ defined for all s except the nonpositive integers. Following the rule derived from integration by parts, it could make sense for $\mathrm{Re}(s) > -1$ to define

$$\Gamma(s) = F_1(s) = \frac{\Gamma(s+1)}{s}.$$

This definition is consistent with the gamma function as usually defined, and we have extended Γ to a meromorphic function on the half-plane $\mathrm{Re}(s) > 1$ with one pole at $s = 0$. In general now, define

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots(s+1)s}.$$

This is meromorphic for $\text{Re}(s) > -m$ with simple poles for $s = -m+1, -m+2, \ldots, -1, 0$. The residues at these poles are

$$\operatorname*{Res}_{s=-n} F_m(s) = \frac{\Gamma(m-n)}{(m-n-1)!(-1)(-2)\cdots(-n)} = \frac{(-1)^n}{n!}.$$

We can see that for all m, F_m agrees with Γ when $\text{Re}(s) > 0$. By the uniqueness of analytic continuation, it must be that whenever $-m < \text{Re}(s) < -m+1$ that the unique value of any analytic continuation of Γ at s must be $F_m(s)$, and thus we have obtained the analytic continuation for Γ.

We will now derive a functional equation for Γ. This requires first establishing the following result:

Lemma 2.1. *For* $0 < a < 1$,

$$\int_0^\infty \frac{v^{a-1}}{1+v}\,dv = \frac{\pi}{\sin(a\pi)}.$$

Proof. Notice that with the substitution $v = e^x$, we obtain

$$\int_0^\infty \frac{v^{a-1}}{1+v}\,dv = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx.$$

Therefore, it suffices to show that

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin(a\pi)}.$$

This will be done through the use of contour integration. Consider a contour γ_R which looks like the following:

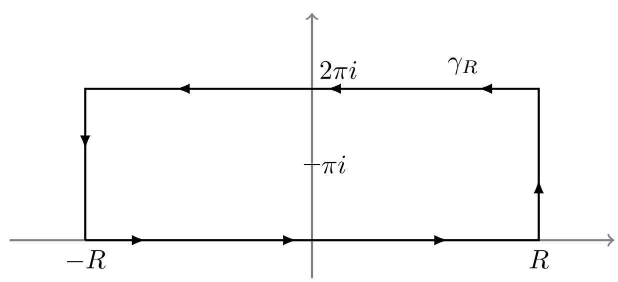

Figure 1: The contour

We will integrate $f(z) = \frac{e^{az}}{1+e^z}$ over γ_R. f is meromorphic, with singularities whenever $e^z = -1$, so $z = \pi i + 2\pi k$ for $k \in \mathbb{Z}$. The only such z in the interior of our region is $z = \pi i$, so this is the only point at which we need to compute a residue. We find that

$$\operatorname*{Res}_{z=\pi i}\left(\frac{e^{az}}{1+e^z}\right) = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}.$$

So, by the residue theorem, we have

$$I_{\gamma_R} = \oint_{\gamma_R} \frac{e^{az}}{1+e^z}\, dz = -2\pi i e^{a\pi i}.$$

Write

$$I_{\gamma_R} = I_r(R) + I_u(R) + I_\ell(R) + I_d(R),$$

where $I_r(R)$ is the integral along the real line

$$I_r(R) = \int_{-R}^{R} f(z)\, dz \xrightarrow[R\to\infty]{} I,$$

$I_u(R)$ is the integral over $\{R+it : 0 \le t \le 2\pi\}$ (upwards), $I_\ell(R)$ is the integral over $\{-t+2\pi i : -R \le t \le R\}$ (leftwards), and $I_d(R)$ is the integral over $\{-R+(2\pi-t)i : 0 \le t \le 2\pi\}$ (downwards). Using the parameterization over the upwards line, we see

$$\begin{aligned} |I_u(R)| &\le \int_0^{2\pi} \left| \frac{e^{a(R+it)}}{1+e^{R+it}} \right| dt \le \int_0^{2\pi} \left| \frac{e^{aR}}{1+e^{R+it}} \right| dt \le \int_0^{2\pi} \left| \frac{e^{aR}}{e^R - 1} \right| dt \\ &\le \frac{2\pi e^{aR}}{e^R-1} = \frac{2\pi e^{(a-1)R}}{1-e^{-R}} \le 2\pi e^{(a-1)R}. \end{aligned}$$

This quantity tends to 0 as $R \to \infty$, as $-1 < a-1 < 0$. Similarly, $I_d(R) \xrightarrow[R\to\infty]{} 0$. For $I_\ell(R)$, we can see that

$$\begin{aligned} I_\ell(R) &= -\int_{-R}^{R} \frac{e^{a(2\pi i - t)}}{1+e^{2\pi i - t}}\, dt = -e^{2\pi i a} \int_{-R}^{R} \frac{e^{-at}}{1+e^{-t}}\, dt \\ &= e^{2\pi i a} \int_{R}^{-R} \frac{e^{au}}{1+e^u}\, du = -e^{2\pi i a} I_r(R). \end{aligned}$$

This means that

$$-2\pi i e^{a\pi i} = \lim_{R\to\infty} I_r(R)(1-e^{2\pi i a})$$

so

$$I = \frac{2\pi i e^{a\pi i}}{e^{2\pi i a}-1} = \frac{2\pi i}{e^{\pi i a} - e^{-\pi i a}} = \frac{\pi}{\sin(a\pi)}.$$

□

We obtain the following functional equation as a consequence.

Theorem 2.2. *For all $s \in \mathbb{C} \backslash \mathbb{Z}$,*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Proof. It suffices to show that this holds for $0 < s < 1$. Then, by analytic continuation, it should hold on all of $\mathbb{C}$. Notice that

$$\Gamma(1-s) = \int_0^\infty e^{-u} u^{-s}\, du = t \int_0^\infty e^{-vt} (vt)^{-s}\, dv$$

where the change of variable $u \mapsto tv$ was made for some $t > 0$. With this change of variable, it can be seen that

$$\begin{aligned} \Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-t} t^{s-1} \Gamma(1-s)\, dt = \int_0^\infty e^{-t} t^s \left(\int_0^\infty e^{-vt} (vt)^{-s}\, dv \right) dt \\ &= \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s}\, dv\, dt = \int_0^\infty \frac{v^{-s}}{1+v}\, dv = \frac{\pi}{\sin(\pi s)} \end{aligned}$$

(as all terms are positive, the order of integration was able to be switched to arrive at the last integral, and the identity $\sin(\pi(1-s)) = \sin(\pi s)$ was used). □

As a consequence, $1/\Gamma$ is entire with simple zeros at the nonpositive integers.

3 The zeta function

The zeta function is initially defined for real $s > 1$ by

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.$$

Note that the sum is convergent whenever $\mathrm{Re}(s) > 1$, and it is in fact holomorphic in this region (as a consequence of uniform convergence).

3.1 Analytic continuation

To analytically continue ζ, we will need to see its relation to Γ. Before this, we first consider a new function, the theta function.

Definition 3.1. *The theta function is defined for real $t > 0$ by*

$$\vartheta(t) = \sum_{n=-\infty}^\infty e^{-\pi n^2 t}.$$

Theorem 3.2. *The theta function satisfies the functional equation*

$$\vartheta(t) = t^{-1/2} \vartheta(1/t).$$

Proof. Let $f_t(x) = e^{-\pi x^2 t}$. Computing the Fourier transform of f_t, we see

$$\begin{aligned} \widehat{f_t}(x) &= \int_{-\infty}^\infty e^{-\pi x^2 t} e^{2\pi i \xi x}\, dx = \int_{-\infty}^\infty e^{-\pi t\left(x^2 - \frac{2i\xi}{t} x\right)}\, dt \\ &= \int_{-\infty}^\infty e^{-\pi t\left(x - \frac{i\xi}{t}\right)^2 - \frac{\pi}{t}\xi^2}\, dx = \frac{1}{\sqrt{t}} e^{-\pi \xi^2 / t}. \end{aligned}$$

By the Poisson summation formula (1.3) and Fubini's theorem, we have

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \sum_{n=-\infty}^{\infty} t^{-1/2} e^{-\pi \xi^2/t} = t^{-1/2}\vartheta(1/t).$$

□

By this functional equation, ϑ is $O(t^{-1/2})$ around $t = 0$. Additionally,

$$|\vartheta(t) - 1| = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t} \leq 2\sum_{n=1}^{\infty} e^{-\pi n t} \leq Ce^{-\pi t}$$

By these bounds, a infinite sum-integral interchange is again justified by Fubini's theorem when showing the following result:

Lemma 3.3. *If* $\mathrm{Re}(s) > 1$*, then*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^{\infty} u^{s/2-1}(\vartheta(u) - 1)\, du.$$

Proof. Consider the integral

$$\int_0^{\infty} e^{-\pi n^2 u} u^{s/2-1}\, du.$$

Make a substitution $t = \pi n^2 u$ to get

$$\int_0^{\infty} e^{-\pi n^2 u} u^{s/2-1}\, du = (\pi n^2)^{-s/2}\int_0^{\infty} e^{-t} t^{s/2-1}\, dt = \pi^{-s/2}\Gamma(s/2) n^{-s}.$$

Notice that

$$\frac{\vartheta(u) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 u}$$

so

$$\frac{1}{2}\int_0^{\infty} u^{s/2-1}(\vartheta(u) - 1)\, du = \int_0^{\infty} u^{s/2-1}\sum_{n=1}^{\infty} e^{-\pi n^2 u}\, du = \sum_{n=1}^{\infty}\int_0^{\infty} u^{s/2-1} e^{-\pi n^2 u}\, du$$

$$= \sum_{n=1}^{\infty} \pi^{-s/2}\Gamma(s/2) n^{-s} = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

□

Now, we define the xi function for $\mathrm{Re}(s) > 1$ by

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Theorem 3.4. *The xi function is holomorphic for* $\mathrm{Re}(s) > 1$. *Additionally, it has an analytic continuation to all of* $\mathbb{C}$ *as a meromorphic function with simple poles at* $s = 0$ *and* $s = 1$*, and it satisfies the functional equation*

$$\xi(s) = \xi(1-s)$$

for all $s \in \mathbb{C}$.

Proof. Let $\psi(u) = (\vartheta(u) - 1)/2$. By the functional equation for ϑ (3.2),

$$\psi(u) = u^{-1/2}\psi(1/u) + \frac{u^{-1/2}}{2} - \frac{1}{2}$$

Using 3.3, we have

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^\infty u^{s/2-1}\psi(u)\,du \\
&= \int_0^1 u^{s/2-1}\psi(u)\,du + \int_1^\infty u^{s/2-1}\psi(u)\,du \\
&= \int_0^1 u^{s/2-1}\left(u^{-1/2}\psi(1/u) + \frac{u^{-1/2}}{2} - \frac{1}{2}\right)du + \int_1^\infty u^{s/2-1}\psi(u)\,du \\
&= \int_0^1 \frac{1}{u}\left(u^{(s-1)/2}\psi(1/u) + \frac{u^{(s-1)/2}}{2} - \frac{u^{s/2}}{2}\right)du + \int_1^\infty u^{s/2-1}\psi(u)\,du \\
&= \frac{1}{s-1} - \frac{1}{s} + \int_0^1 u^{(s-1)/2}\frac{\psi(1/u)}{u}\,du + \int_1^\infty u^{s/2-1}\psi(u)\,du.
\end{aligned}$$

Making a substitution $t = 1/u$ on the first remaining integral gives

$$\int_0^1 u^{(s-1)/2}\frac{\psi(1/u)}{u}\,du = -\int_\infty^1 t^{(1-s)/2}\frac{\psi(t)}{t}\,dt = \int_1^\infty t^{(1-s)/2}\frac{\psi(t)}{t}\,dt$$

so

$$\begin{aligned}
\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \int_0^\infty u^{s/2-1}\psi(u)\,du \\
&= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{(1-s)/2} + u^{s/2}\right)\frac{\psi(u)}{u}\,du.
\end{aligned}$$

Thus, for $\mathrm{Re}(s) > 1$ we can write

$$\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{(1-s)/2} + u^{s/2}\right)\frac{\psi(u)}{u}\,du.$$

The integral term converges for all $s \in \mathbb{C}$ since $\psi(u)$ decreases with an exponential upper bound as $u \to \infty$, so this is defined and holomorphic everywhere except $s = 0$ and $s = 1$ (at these points, ξ has simple poles). $\square$

Corollary 3.4.1. *The zeta function has a meromorphic continuation into the entire complex plane with one singularity as a simple pole at* $s = 1$.

Proof. Recall the definition of the xi function, and consider

$$\zeta(s) = \frac{\pi^{s/2}\xi(s)}{\Gamma(s/2)}.$$

$1/\Gamma(s/2)$ is entire, and the simple pole of ξ at $s = 0$ is canceled by the zero of $1/\Gamma(s/2)$ at $s = 0$. Only the $s = 1$ pole remains. □

3.2 Relation to prime numbers

The ζ function is related to prime numbers, as indicated by the Euler product:

Theorem 3.5 (Euler product for the zeta function)**.** *If* $\mathrm{Re}(s) > 1$*, then*

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

Proof. Observe that by geometric series,

$$\frac{1}{1 - p^{-s}} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}} + \cdots$$

We will need this in the proof of the identity. Suppose that we have two positive integers M and N, where $M > N$. By the fundamental theorem of arithmetic, we can express any $n \leq N$ by a unique prime factorization. Each prime in this factorization must be at most N and repeated less than M times, so

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n^s} \leq \prod_{p \leq N} \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right) \leq \prod_{p \leq N} \left(\frac{1}{1 - p^{-s}}\right) \\ \leq \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s}}\right)\end{aligned}$$

We let N tend to infinity to obtain one direction of the inequality:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leq \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s}}\right)$$

and of course, the product on the right converges as

$$\sum_{p \text{ prime}} \left|\frac{1}{1 - p^{-s}} - 1\right| = \sum_{p \text{ prime}} \left|\frac{1}{p^s - 1}\right| \leq \sum_{n=1}^{\infty} \left|\frac{1}{n^s - 1}\right| < \infty$$

To show the reverse direction on the right (which would show the product converges anyway), we use the funamdental theorem of arithmetic to write

$$\prod_{p \leq N} \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{Ms}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Then, if we take $M \to \infty$ we obtain

$$\prod_{p \leq N} \left(\frac{1}{1-p^{-s}} \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}$$

and we take $N \to \infty$ to complete the proof. $\square$

One other way we can see this result is by directly appealing to the fundamental theorem of arithmetic. First, notice that by geometric series,

$$\sum_{k=1}^{\infty} p^{-ks} = \frac{1}{1-p^{-s}}.$$

Using this, we may write the Euler product as

$$\prod_{p \text{ prime}} \sum_{k=1}^{\infty} p^{-ks} = (1+2^{-s}+2^{-2s}+\dots)(1+3^{-s}+3^{-2s}+\dots)(1+5^{-s}+5^{-2s}+\dots)\cdots$$

If we pick any natural number n, then n^{-s} must appear in this product exactly once, by the existence and uniqueness of prime factorization. This means that we have a sum of n^{-s} for all natural numbers, "establishing" the equivalence

$$\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

3.3 Locations of zeros

Definition 3.6 (Critical strip)**.** *The critical strip is* $\{s : 0 \leq \mathrm{Re}(s) \leq 1\}$.

This strip takes on an important role in studying the zeros of ζ:

Theorem 3.7. *The only zeros of ζ outside the critical strip are all the negative even integers,* $s = -2, -4, -6, \dots$.

Proof. Recall 3.4.1, which lets us write

$$\zeta(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s).$$

Notice that for $\mathrm{Re}(s) < 0$, the following hold:

1. $\zeta(1-s)$ has no zeros, as $\mathrm{Re}(1-s) > 1$
2. $\Gamma((1-s)/2)$ also has no zeros
3. $1/\Gamma(s/2)$ has zeros whenever $s/2$ is a negative integer (so s is a negative even integer)

This, combined with the known fact that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$, shows the desired result. $\square$

Zeros in the critical strip are known as *non-trivial zeros*; the negative integers are the *trivial zeros*. The well-known Riemann hypothesis proposes an even stronger statement about such zeros: namely, that all non-trivial zeros of ζ have real part equal to 1/2.

3.3.1 No zeros with real part 1

We have already shown at all nontrivial solutions to $\zeta(s) = 0$ lie in the critical strip $0 \leq \text{Re}(s) \leq 1$. We now rule out $\text{Re}(s) = 1$, as done in Theorem 1.2 on page 185 of Stein and Shakarchi.

To show this, we will rely on the following two identities, which I will not include the proofs for here:

1. If $\text{Re}(s) > 1$, then

$$\log \zeta(s) = \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}$$

   where $c_n = 1/m$ if $n = p^m$ and $c_n = 0$ otherwise.

2. For $\theta \in \mathbb{R}$, $3 + 4\cos\theta + \cos 2\theta \geq 0$.

As a result, we have

$$\log |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 0.$$

Proof. If we write $s = \sigma + it$, then

$$\text{Re}(n^{-s}) = \text{Re}(n^{-\sigma}n^{-it}) = n^{-\sigma}\cos(t \log n)$$

and thus

$$\begin{aligned}
&\log |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \\
&= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + it)| + \log|\zeta(\sigma + 2it)| \\
&= 3\,\text{Re}[\log\zeta(\sigma)] + 4\,\text{Re}[\log\zeta(\sigma + it)] + \text{Re}[\zeta(\sigma + 2it)] \\
&= \sum_n c_n n^{-\sigma}(3 + 4\cos\theta_n + \cos 2\theta_n) \geq 0
\end{aligned}$$

where $\theta_n = t \log n$. □

As a consequence, $\zeta(s)$ is zero-free for $\text{Re}(s) = 1$. This is seen by using holomorphicity at $1 + it$ for real $t \neq 0$ and the fact that logarithms of real numbers between 0 and 1 are negative. Thus, we establish the theorem

Theorem 3.8. $\zeta(1 + it) \neq 0$ *for all* $t \in \mathbb{R}$.

3.4 Estimates for $1/\zeta$

Theorem 3.9. *For every $\epsilon > 0$, $1/|\zeta(s)| \le c_\epsilon |t|^\epsilon$ when $s = \sigma + it$, $\sigma \ge 1$, and $|t| \ge 1$.*

The proof of this theorem can be found on pages 187-188 of Stein and Shakarchi (Prop. 1.6.).

4 Proof of the prime number theorem

4.1 Definitions of some functions

Definition 4.1 (Prime counting function)**.** *The function π is known as the prime-counting function, and*

$$\pi(x) = \sum_{p \le x} 1 = \text{number of primes } \le x.$$

As its name suggests, it counts primes.

4.1.1 Some auxiliary functions

Chebyshev introduced a ψ function:

Definition 4.2 (Psi function)**.** *Chebyshev's ψ function is defined by*

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{p \le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p.$$

The behavior of this function can be used to make statements about $\pi(x)$, while also being a lot easier to work with analytically.

Definition 4.3 (Von Mangoldt function)**.** *The Von Mangoldt function is defined by*

$$\Lambda(n) = \begin{cases} p & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using the Von Mangoldt function, we may write

$$\psi(x) = \sum_{1 \le n \le x} \Lambda(n).$$

4.2 A sequence of propositions

In this section, a sequence of propositions are stated. These are used to eventually build a proof of the prime number theorem (Section 4.4):

Theorem 4.4 (Prime number theorem)**.** *The prime counting function π satisfies*

$$\lim_{n\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

For notational convenience, we define the following:

Definition 4.5 (Asympototic)**.** *Two functions $f(x)$ and $g(x)$ on $\mathbb{R}$ are said to be asymptotic as $x \to \infty$ if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

We write this as $f(x) \sim g(x)$.

With this notation, we may write the prime number theorem as $\pi(x) \sim x/\log x$.

Proposition 4.6. *If $\psi(x) \sim x$, then $\pi(x) \sim x/\log x$.*

Proof. We have that

$$\psi(x) = \sum_{p\le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \le \sum_{p \le x} \log x = \pi(x) \log x$$

and thus

$$1 = \lim_{x\to\infty} \frac{\psi(x)}{x} \le \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x}.$$

For the other direction, fix an arbitrary $\alpha \in (0,1)$. Then,

$$\psi(x) \ge \sum_{p\le x} \log p \ge \sum_{x^\alpha < p \le x} \log p \ge (\pi(x) - \pi(x^\alpha)) \log x^\alpha$$

from which it follows that

$$\psi(x) + \alpha\pi(x^\alpha) \log x \ge \alpha\pi(x) \log x.$$

Dividing both sides by x, we have

$$\frac{\psi(x)}{x} + \alpha \frac{\pi(x^\alpha)\log x}{x} \ge \alpha \frac{\pi(x)}{x/\log x}.$$

Since $\pi(x) \le x$, we have

$$\alpha \frac{\pi(x)}{x/\log x} \le \frac{\psi(x)}{x} + \alpha x^{\alpha-1} \log x$$

but $\alpha - 1 \in (-1, 0)$, so this term vanishes as $x \to \infty$, and thus

$$\limsup_{x\to\infty} \alpha \frac{\pi(x)}{x/\log x} \le \lim_{x\to\infty} \frac{\psi(x)}{x} = 1.$$

Since $\alpha < 1$, we have

$$\limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \le 1$$

completing the proof. ∎

As an additional remark, this is an if and only if condition. The other direction can be shown similarly, but is not needed for our purposes.

The next proposition follows as a consequence. First, we define the function ψ_1 by

$$\psi_1(x) = \int_1^x \psi(u)\, du.$$

Note that since $\psi(u) = 0$ for $0 \leq u \leq 1$, equality holds if we replace the lower limit of integration with 0.

It turns out the following is true:

Proposition 4.7. *If $\psi_1(x) \sim x^2/2$, then $\pi(x) \sim x/\log x$.*

This follows from the fact that $\psi_1(x) \sim x^2/2$ implies that $\psi(x) \sim x$; this statement may seem rather intuitive, and an interested reader can find the proof of it on page 190 of Stein and Shakarchi. By 4.6, this would further imply that $\pi(x) \sim x/\log x$.

4.3 A closer look at ψ_1

We can relate ψ_1 with ζ. Recall that

$$\log \zeta(s) = \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{p^{-ms}}{m}.$$

Differentiating to get the logarithmic derivative of ζ, we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{p \text{ prime}} \sum_{m=1}^{\infty} (\log p) p^{-ms} = -\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}.$$

We can switch the signs of each side, providing us with something that will be useful:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \tag{1}$$

Proposition 4.8. *For all $c > 1$,*

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds.$$

One may notice that this integral seems to be an inverse Mellin transform. This is no coincidence, and can be studied further in this manner if desired.

Lemma 4.9. *If $c > 0$, then*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}\, ds = \begin{cases} 0 & 0 < a \leq 1 \\ 1 - 1/a & 1 \leq a \end{cases}$$

This integral is taken over the vertical line $\mathrm{Re}(s) = c$, and converges as $|a^s| = a^c$.

To evaluate this integral, we consider a semicircular contour centered at $(c, 0)$ with a vertical line segment along $\mathrm{Re}(s) = c$ and an arc going counterclockwise from (c, r) to $(c, -r)$. For sufficiently large r, the sum of the residues is $1 - 1/a$, and so it remains to show that the part along the arc vanishes as $r \to \infty$ (which is done using a standard ML inequality argument). These steps prove the desired result for $a \geq 1$.

For $0 < a < 1$, we consider a similar contour but it is on the right side, going counterclockwise from $(c, -r)$ to (c, r). Since there are no poles inside this contour, the sum of residues is zero there.

Now we may prove 4.8. Let $f_n(u) = 1$ if $n \leq u$ and 0 otherwise. Then

$$\psi(u) = \sum_{n=1}^{\infty} \Lambda(u) f_n(u).$$

so, by definition of ψ_1,

$$\psi_1(x) = \int_0^x \sum_{n=1}^{\infty} \Lambda(u) f_n(u) = \sum_{n=1}^{\infty} \int_0^x \Lambda(n) f_n(u)\, du = \sum_{n \leq x} \Lambda(n)(x - n),$$

and then using (1) and 4.9 we have

$$\begin{aligned}
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} x n^s \frac{(x/n)^s}{s(s+1)} \frac{\Lambda(n)}{n^s}\, ds \\
&= \sum_{n=1}^{\infty} \Lambda(n) x \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)}\, ds \\
&= \sum_{n=1}^{\infty} \Lambda(n) x \left(1 - \frac{n}{x}\right) = \sum_{n=1}^{\infty} \Lambda(n)(x - n) \\
&= \psi_1(x),
\end{aligned}$$

proving 4.8.

4.3.1 Riemann's explicit formula

This relates to **Riemann's explicit formula**,

$$\begin{aligned}
\psi(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds \\
&= x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2} \log(1 - x^{-2})
\end{aligned}$$

for any x where $\psi(x)$ is equal to its left-side limit (so $x \neq 2, 3, 4, 5, 7, 8, 9, 11, \ldots$), where

$$\sum_{\rho} \frac{x^\rho}{\rho} = \lim_{T \to \infty} \sum_{\mathrm{Im}(\rho) \leq T} \frac{x^\rho}{\rho}$$

with ρ referring to non-trivial zeros of the zeta function. The first equality follows by differentiation, and the second from an application of the residue theorem, noting that $\zeta'(0)/\zeta(0) = \log(2\pi)$.

The later two terms in Riemann's explicit formula are comparatively small, and can be written as $O(1)$. Also notice that this formula relates the location of non-trivial zeros of ζ to a bound on $|\psi(x) - x|$. This suggests that the bound on the error of the prime number theorem, $\pi(x) \sim x/\log x$, relates to the locations of these zeros. If true, the Riemann hypothesis would provide the strongest possible such bound.

4.4 Proving the main result

We are now at a position where we are ready to prove the prime number theorem in a few steps. Fix some $c > 1$, and let $F(s)$ denote the (familiar) integrand

$$F(s) = \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right).$$

As per the treatment in Stein and Shakarchi, we will use the following:

- 4.8, relating ψ_1 to ζ using the integrand $F(s)$,
- 3.8, which states $\zeta(1 + it) \neq 0$ for all $t \in \mathbb{R}$, as well as the estimates provided by 3.9 and the following:

  Proposition 4.10. *Suppose $s = \sigma + it$ where $\sigma, t \in \mathbb{R}$. Then, for each σ_0 satisfying $0 \leq \sigma_0 \leq 1$ and every $\epsilon > 0$, there exists a constant c_ϵ so that*

  (i) $|\zeta(s)| \leq c_\epsilon |t|^{1-\sigma_0+\epsilon}$ *if* $\sigma_0 \leq \sigma$ *and* $|t| \geq 1$,

  (ii) $|\zeta'(s)| \leq c_\epsilon |t|^\epsilon$ *if* $1 \leq \sigma$ *and* $|t| \geq 1$.

  (proposition 2.7 from Chapter 6 of Stein and Shakarchi, found on page 173).

Recall that

$$\psi_1(x) = \int_{c-i\infty}^{c+i\infty} F(s)\, ds.$$

We may deform the line $\mathrm{Re}(s) = c$ into a new contour while keeping the integral the same, as shown in the following figure. $T \geq 3$ will be chosen appropriately large later. The curve $\gamma(T)$ has real part 1 for $|\,\mathrm{Im}(s)| = |t| \geq T$, and stretches horizontally to reach the $\mathrm{Re}(s) = c$ line before returning later.

The lack of poles of F when $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s) = 1$, $\mathrm{Im}(s) \neq 0$, allows us to show by the Cauchy integral theorem that

$$\psi_1(x) = \int_{\gamma(T)} F(s)\, ds.$$

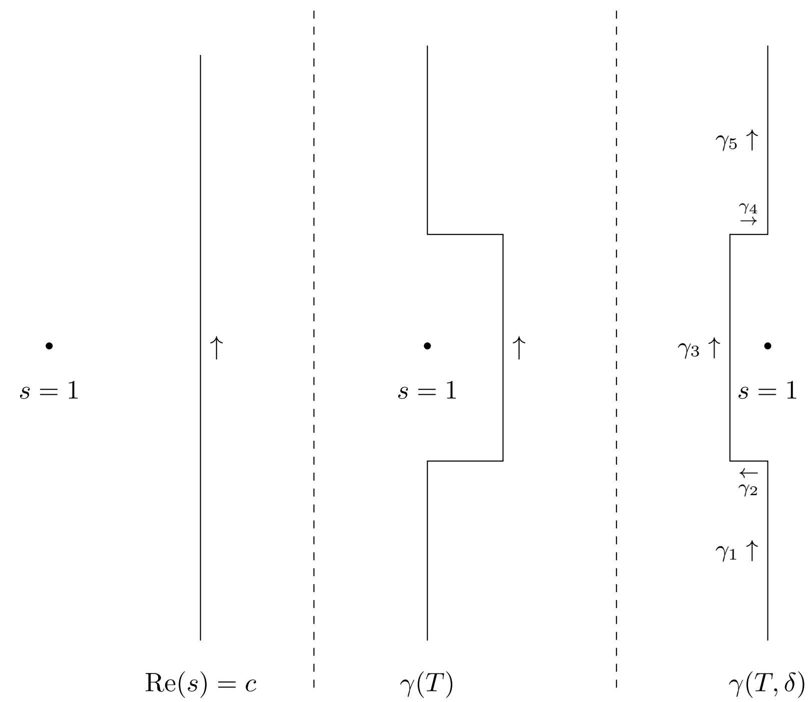

Figure 2: The original line, the path $\gamma(T)$, and the path $\gamma(T, \delta)$

We can also make use of another contour, $\gamma(T, \delta)$, as detailed in the figure. Again, here the parts lying along $\mathrm{Re}(s) = 1$ (so, γ_1 and γ_5) have $|t| \geq T$. Along γ_2 and γ_4, which have constant imaginary part $\pm T$, $1 - \delta \leq \mathrm{Re}(s) \leq 1$.

Now, we consider going from $\gamma(T)$ to $\gamma(T, \delta)$, where δ is chosen to be small enough so that there are no zeros of ζ in $\{\sigma + it : 1 - \delta < \sigma < 1, |t| < T\}$ (which is possible by continuity and the fact that ζ has a pole at $s = 1$).

By applying the argument principle to ζ'/ζ, it can be seen that

$$\operatorname*{Res}_{s=1} F(s) = \frac{x^2}{2}.$$

Thus,

$$\frac{1}{2\pi i} \int_{\gamma(T)} F(s)\, ds = \frac{x^2}{2} + \frac{1}{2\pi i} \int_{\gamma(T,\delta)} F(s)\, ds.$$

It remains to bound F over $\gamma(T, \delta)$, which is decomposed into $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$. Along γ_j for $j = 1, 5$, we have $|x^{1+s}| = x^{1+1} = x^2$, $|\zeta'(s)/\zeta(s)| = O(\sqrt{|t|})$, so

$$\left| \int_{\gamma_j} F(s)\, ds \right| \leq C x^2 \int_T^{\infty} |t|^{-3/2}\, dt < \frac{\epsilon}{4} x^2$$

for sufficiently large T. On γ_3, notice that $|x^{s+1}| = x^{1+1-\delta} = x^{2-\delta}$ and thus, depending on T, we have some constant such that

$$\left|\int_{\gamma_3} F(s)\, ds\right| \leq C_T x^{2-\delta}.$$

For the horizontal segments γ_j $(j = 2, 4)$, we have (for some $C'_T > 0$) that

$$\left|\int_{\gamma_j} F(s)\, ds\right| \leq C'_T \int_{1-\delta}^{1} x^\sigma \, d\sigma \leq C'_T \frac{x^2}{\log x}.$$

Recalling that

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, ds = \frac{1}{2\pi i} \int_{\gamma(T)} F(s)\, ds,$$

we have by the triangle inequality that

$$\left|\psi_1(x) - \frac{x^2}{2}\right| \leq \frac{\epsilon}{4} x^2 + C_T x^{2-\delta} + 2C'_T \frac{x^2}{\log x}.$$

Now, dividing both sides by $x^2/2$, we obtain

$$\left|\frac{\psi_1(x)}{x^2/2} - 1\right| \leq \frac{\epsilon}{2} + 2C_T x^{-\delta} + 4C'_T \frac{1}{\log x}.$$

For sufficiently large x, we can make $2C_T x^{-\delta}$ and $4C'_T/\log x$ sufficiently small (less than $\epsilon/4$ each), and thus

$$\left|\frac{\psi_1(x)}{x^2/2} - 1\right| < \epsilon$$

Since the choice of $\epsilon > 0$ is arbitrary, this means that

$$\lim_{x\to\infty} \frac{\psi_1(x)}{x^2/2} = 1$$

or, using tilde notation, that $\psi_1(x) \sim x^2/2$. Through 4.7, we obtain

Theorem 4.4 (Prime number theorem). *The prime counting function π satisfies*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

This completes the proof of the prime number theorem.

We now look at some consequences of the prime number theorem.

5 A look at some consequences

The following corollary gives us an asymptotic for the n-th prime number, p_n.

Corollary 5.0.1. *If p_n denotes the n-th prime number, p_n satisfies*

$$p_n \sim n \log n.$$

For this, we need the following lemma.

Lemma 5.1. *Suppose $f(n) \sim g(n)$ and that there exists constants $N \in \mathbb{N}$ and $M > 1$ such that for $n \geq N$, $g(n) \geq M$. Then, $\log f(n) \sim \log g(n)$.*

Proof. By assumption, $f(n) \sim g(n)$, so

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

Thus, if we choose some $\epsilon > 0$ such that $\epsilon \leq A$ (where $A < 1$ is fixed), there exists an $N_1 \in \mathbb{N}$ such that for $n \geq N$,

$$1 - \epsilon < \frac{f(n)}{g(n)} < 1 + \epsilon$$

Multiplying the inequality by $g(n)$ and taking a logarithm, we have

$$\log(1-\epsilon) + \log g(n) < \log f(n) < \log(1+\epsilon) + \log g(n)$$

Therefore, we have

$$1 + \frac{\log(1-\epsilon)}{\log g(n)} < \frac{\log f(n)}{\log g(n)} < 1 + \frac{\log(1+\epsilon)}{\log g(n)}$$

Applying the inequality $\log(1+x) \leq x$ and using that for $0 < \epsilon < A$,

$$\log(1-\epsilon) \geq -\frac{\log(1-A)}{A}\epsilon,$$

we obtain

$$1 - \frac{2\epsilon}{A \log g(n)} < \frac{\log f(n)}{\log g(n)} < 1 + \frac{\epsilon}{\log g(n)},$$

and $\log g(n) \geq \log M$ when $n \geq N_2$ for some $N_2 \in \mathbb{N}$, so with $n \geq \max\{N_1, N_2\}$ we have

$$1 - \frac{2\epsilon}{A \log M} < \frac{\log f(n)}{\log g(n)} < 1 + \frac{\epsilon}{\log M}.$$

This effectively completes the proof. □

Exercise 1. There is a much simpler proof of this result. Find it!

Using this lemma, we may prove that $p_n \sim n \log n$, where p_n is the n-th prime number, as a consequence of the prime number theorem.

Proof. By the prime number theorem, we have $\pi(x) \sim x/\log x$. Since $\lim_{x\to\infty} x/\log x = \infty$, we may apply 5.1 to obtain $\log \pi(x) \sim \log(x/\log x)$, and $\log(x/\log x) = \log x - \log\log x$. As a consequence, we have

$$\frac{\log \pi(x)}{\log x} \sim 1 - \frac{\log\log x}{\log x} \sim 1$$

Thus, if we let $x = p_n$, we have

$$\frac{\log n}{\log p_n} \sim 1, \quad \text{or} \quad \log n \sim \log p_n.$$

since $\pi(p_n) = n$. Turning back to the prime number theorem, we have $n \sim p_n/\log p_n$. So, we can compute

$$\lim_{n\to\infty} \frac{p_n}{n\log n} = \lim_{n\to\infty} \frac{p_n}{n\log p_n}\frac{\log p_n}{\log n} = (1)(1) = 1$$

showing that $p_n \sim n\log n$. □

Another result is another asymptotic relation for the prime counting function.

Definition 5.2 (Logarithmic integral)**.** *The logarithmic integral is the function* Li *given by*

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t}\,dt.$$

Corollary 5.2.1. *We have that*

$$\pi(x) \sim \mathrm{Li}(x).$$

Proof. Since $\sim$ is an equivalence relation, it suffices to show

$$\frac{x}{\log x} \sim \mathrm{Li}(x).$$

Let us proceed in this manner. By the use of L'Hopital's rule twice, we find

$$\begin{aligned}\lim_{x\to\infty} \frac{\mathrm{Li}(x)}{x/\log x} = \lim_{x\to\infty} \frac{\mathrm{Li}(x)\log x}{x} = \lim_{x\to\infty}\left(1 + \frac{1}{x}\mathrm{Li}(x)\right) = 1 + \lim_{x\to\infty}\frac{\mathrm{Li}(x)}{x} \\ = 1 + \lim_{x\to\infty}\frac{1}{\log x} = 1.\end{aligned}$$

□

It turns out that this provides a better approximation to $\pi(x)$ than $x/\log x$, and that following Riemann's explicit formula, the Riemann hypothesis is equivalent to

$$\mathrm{Li}(x) = \pi(x) + O(x^{1/2}\log x).$$

Exercise 2. Verify that $\mathrm{Li}(x) = \pi(x) + O(x^{1/2}\log x)$.